

What a Classical r -Matrix Really Is

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*To my friend and colleague K.C. Reddy
on occasion of his retirement.*

Abstract

The notion of classical r -matrix is re-examined, and a definition suitable to differential (-difference) Lie algebras, – where the standard definitions are shown to be deficient, – is proposed, the notion of an \mathcal{O} -operator. This notion has all the natural properties one would expect from it, but lacks those which are artifacts of finite-dimensional isomorphisms such as not true in differential generality relation $\text{End}(V) \simeq V^* \otimes V$ for a vector space V . Examples considered include a quadratic Poisson bracket on the dual space to a Lie algebra; generalized symplectic-quadratic models of such brackets (aka Clebsch representations); and Drinfel'd's 2-cocycle interpretation of nondegenerate classical r -matrices.

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1 From Artin relation to Quantum Yang–Baxter equation to Classical Yang–Baxter equation

Hier ist kein Warum.

This paper is written with a non-expert in mind, and the text is purposely self-contained apart from a few references to basic properties of the algebraic calculus of variations and Hamiltonian formalism. In this section we derive the Classical Yang–Baxter equation (CYBE) as the quasiclassical limit of the Quantum Yang–Baxter equation (QYBE); the latter will be seen in a moment as a special form of the Artin relation for the generators of the braid group.

We thus start from a purely finite-dimensional view-point; finite-dimensional Lie algebras will come in later on through an interpretation of the CYBE, and differential Lie algebras will appear later still. Have I mentioned that most, if not all, of the results in this Section are gleamed from the confidential list of examination questions given annually to all low-level NSA employees?

Let's fix a vector space V and let

$$S : V \otimes V \rightarrow V \otimes V \quad (1.1)$$

be an operator. S induces the operators $S^{12} = S \otimes \mathbf{1}$ and $S^{23} = \mathbf{1} \otimes S$ acting on $V \otimes V \otimes V$ in an obvious way. The operator S^{13} acts on $V \otimes V \otimes V$ in an equally natural way:

$$S^{13}(e_i \otimes e_j \otimes e_k) = \sum_{ab} S_{ik}^{ab} e_a \otimes e_j \otimes e_b, \quad (1.2)$$

where

$$S(e_i \otimes e_j) = \sum_{cd} S_{ij}^{cd} e_c \otimes e_d, \quad (1.3)$$

and (e_i) is a basis in V .

Denote by $P : V \otimes V \rightarrow V \otimes V$ the permutation operator,

$$P(v_1 \otimes v_2) = v_2 \otimes v_1, \quad \forall v_1, v_2 \in V, \quad (1.4)$$

and let $\mathcal{M} : V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ be the operator of mirror symmetry:

$$\mathcal{M}(v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_2 \otimes v_1. \quad (1.5)$$

It's immediate to see by inspection that

$$\mathcal{M} = P^{23} P^{12} P^{23} = P^{12} P^{23} P^{12}. \quad (1.6)$$

The Artin equation for an arbitrary operator $S : V \otimes V \rightarrow V \otimes V$ is then

$$S^{23} S^{12} S^{23} = S^{12} S^{23} S^{12}. \quad (1.7)$$

That's all there is to it. We now proceed to massage this equation in various directions. Set

$$S = PR \Leftrightarrow R = PS. \quad (1.8)$$

The Artin equation (1.7) will become then

$$P^{23}R^{23}P^{12}R^{12}P^{23}R^{23} = P^{12}R^{12}P^{23}R^{23}P^{12}R^{12}. \quad (1.9)$$

Let $A, \mathcal{U}, \mathcal{V}, \mathcal{W}$ be arbitrary operators $V \otimes V \rightarrow V \otimes V$. The following identities are easy to check and are left for the reader to verify:

$$A^{12}P^{23} = P^{23}A^{13}, \quad (1.10a)$$

$$A^{13}P^{23} = P^{23}A^{12}, \quad (1.10b)$$

$$A^{23}P^{12} = P^{12}A^{13}, \quad (1.10c)$$

$$A^{13}P^{12} = P^{12}A^{23}; \quad (1.10d)$$

$$\mathcal{U}^{23}\mathcal{V}^{12}\mathcal{W}^{23} = \mathcal{W}^{12}\mathcal{V}^{23}\mathcal{U}^{12} \quad (1.11)$$

whenever at least two out of three operators $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are equal to P .

Notice that each one of the formulae (1.10) can be taken as an *invariant definition* of the operator A^{13} , thus avoiding the coordinate definition (1.2).

Now let us transform separately each side of the equation (1.9). For the LHS we get

$$\begin{aligned} P^{23}R^{23}P^{12}R^{12}P^{23}R^{23} &\stackrel{[\text{by (1.10c,a)}]}{=} P^{23}P^{12}R^{13}P^{23}R^{13}R^{23} \\ &\stackrel{[\text{by (1.10b)}]}{=} P^{23}P^{12}P^{23}R^{12}R^{13}R^{23} = \mathcal{M}R^{12}R^{13}R^{23}, \end{aligned} \quad (1.12L)$$

while for the RHS of the equation (1.9) we obtain

$$\begin{aligned} P^{12}R^{12}P^{23}R^{23}P^{12}R^{12} &\stackrel{[\text{by (1.10a,c)}]}{=} P^{12}P^{23}R^{13}P^{12}R^{13}R^{12} \\ &\stackrel{[\text{by (1.10d)}]}{=} P^{12}P^{23}P^{12}R^{23}R^{13}R^{12} = \mathcal{M}R^{23}R^{13}R^{12}. \end{aligned} \quad (1.12R)$$

Equating the expressions (1.12L) and (1.12R) we find

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}. \quad (1.13)$$

This is called QYBE. Since Artin equation (1.7) is satisfied by $S = P$, the QYBE equation (1.13) is satisfied by $R = \mathbf{1}$. Let's look for *perturbations* of this solution: set

$$R = \mathbf{1} + hr + h^2\rho + O(h^3) \quad (1.14)$$

with some formal parameter h . Then

$$\begin{aligned} R^{12}R^{13}R^{23} &= (\mathbf{1} + hr^{12} + h^2\rho^{12})(\mathbf{1} + hr^{13} + h^2\rho^{13})(\mathbf{1} + hr^{23} + h^2\rho^{23}) + O(h^3) \\ &= \mathbf{1} + h(r^{12} + r^{13} + r^{23}) + h^2(\rho^{12} + \rho^{13} + \rho^{23}) \\ &\quad + h^2(r^{12}r^{13} + r^{12}r^{23} + r^{13}r^{23}) + O(h^3), \end{aligned} \quad (1.15L)$$

$$\begin{aligned} R^{23}R^{13}R^{12} &= (\mathbf{1} + hr^{23} + h^2\rho^{23})(\mathbf{1} + hr^{13} + h^2\rho^{13})(\mathbf{1} + hr^{12} + h^2\rho^{12}) + O(h^3) \\ &= \mathbf{1} + h(r^{23} + r^{13} + r^{12}) + h^2(\rho^{23} + \rho^{13} + \rho^{12}) \\ &\quad + h^2(r^{23}r^{13} + r^{23}r^{12} + r^{13}r^{12}) + O(h^3). \end{aligned} \quad (1.15R)$$

Comparing the expressions (1.15L) and (1.15R), we see that they differ in h^2 -terms; these yield

$$c(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \quad (1.16)$$

This is called CYBE. As a quasclassical approximation to a noncommutative QYBE (1.13) in an associative framework, CYBE should have Poisson-brackets-related properties and/or interpretations. This is known to be true, and we shall see more of such presently.

The first step in this direction is to realize that since only *commutators* are involved in the CYBE (1.16), the operator r , – called a classical r -matrix, – can be considered not just as an element of the tensor square of the Lie algebra $\text{End}(V)$:

$$\text{End}(V) \otimes \text{End}(V) \approx \text{End}(V \otimes V), \quad (1.17)$$

but as an element of $\mathcal{G} \otimes \mathcal{G}$ for *arbitrary* Lie algebra \mathcal{G} :

$$r \in \mathcal{G} \otimes \mathcal{G}. \quad (1.18)$$

The CYBE (1.16) is then understood as an identity in $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$: If

$$r = \sum_i a_i \otimes b_i, \quad a_i, b_i \in \mathcal{G}, \quad (1.19)$$

then (temporarily stepping outside \mathcal{G} into the Universal enveloping algebra $U(\mathcal{G})$)

$$[r^{12}, r^{13}] = \sum_{ij} [a_i, a_j] \otimes b_i \otimes b_j, \quad (1.20a)$$

$$[r^{12}, r^{23}] = \sum_{ij} a_i \otimes [b_i, a_j] \otimes b_j, \quad (1.20b)$$

$$[r^{13}, r^{23}] = \sum_{ij} a_i \otimes a_j \otimes [b_i, b_j], \quad (1.20c)$$

so that the CYBE (1.16) becomes

$$0 = c \left(\sum_i a_i \otimes b_i \right) = \sum_{ij} ([a_i, a_j] \otimes b_i \otimes b_j + a_i \otimes [b_i, a_j] \otimes a_j \otimes b_j + a_i \otimes a_j \otimes [b_i, b_j]). \quad (1.21)$$

(Finding out the proper Lie-algebraic object to which the CYBE (1.16) in $\mathcal{G}^{\otimes 3}$ is the quasiclassical approximation is far from easy; this is done in Drinfel'd's paper [3].)

Remark 1.22. Had we considered the quasiclassical approximation to the Artin equation (1.7) itself, in the form

$$S = P + h\bar{r} + h^2\bar{\rho} + O(h^3), \quad (1.23)$$

the h^2 -terms would have collected into the equation

$$\bar{r}^{23}\bar{r}^{12}P^{23} + \bar{r}^{23}P^{12}\bar{r}^{23} + P^{23}\bar{r}^{12}\bar{r}^{23} = \bar{r}^{12}\bar{r}^{23}P^{12} + \bar{r}^{12}P^{23}\bar{r}^{12} + P^{12}\bar{r}^{23}\bar{r}^{12}. \quad (1.24)$$

This equation turns into CYBE (1.16) upon the substitution

$$\bar{r} = Pr. \quad (1.25)$$

Let us discuss the skewsymmetry property of the classical r -matrix. If we impose on the operator S the very natural “unitarity” condition

$$S^2 = \mathbf{1}, \quad (1.26)$$

the r -matrix $r = \frac{\partial(PS)}{\partial h} \Big|_{h=0}$ inherits from the unitarity the skewsymmetry condition

$$Pr = -rP. \quad (1.27)$$

The Lie-algebraic r -matrix (1.19) then belongs to $\Lambda^2\mathcal{G}$ rather than $\mathcal{G}^{\otimes 2}$:

$$r = \sum_i a_i \wedge b_i = \sum_i (a_i \otimes b_i - b_i \otimes a_i). \quad (1.28)$$

Although non-skewsymmetric r -matrices play many important rôles in various branches of Mathematics and Physics (see the textbook [1] of Chari and Pressley as the basic reference for what follows), in this paper *all* r -matrices will be considered skewsymmetric, due to the nature of the topics discussed.

For future reference, we shall record the skewsymmetric version of formulae (1.20) for $c(r)$ with the skewsymmetric r -matrix (1.28):

$$[r^{12}, r^{13}] = \sum_{ij} ([a_i, a_j] \otimes b_i \otimes b_j - [a_i, b_j] \otimes b_i \otimes a_j - [b_i, a_j] \otimes a_i \otimes b_j + [b_i, b_j] \otimes a_i \otimes a_j), \quad (1.29a)$$

$$[r^{12}, r^{23}] = \sum_{ij} (a_i \otimes [b_i, a_j] \otimes b_j - a_i \otimes [b_i, b_j] \otimes a_j - b_i \otimes [a_i, a_j] \otimes b_j + b_i \otimes [a_i, b_j] \otimes a_j), \quad (1.29b)$$

$$[r^{13}, r^{23}] = \sum_{ij} (a_i \otimes a_j \otimes [b_i, b_j] - a_i \otimes b_j \otimes [b_i, a_j] - b_i \otimes a_j \otimes [a_i, b_j] + b_i \otimes b_j \otimes [a_i, a_j]). \quad (1.29c)$$

2 Classical r -matrices and 2-cocycles

Any skewsymmetric element $r \in \Lambda^2\mathcal{G}$ satisfying the CYBE (1.16) is called a classical r -matrix.

To every $r \in \mathcal{G} \otimes \mathcal{G}$ we can associate an operator $\mathcal{O} = \mathcal{O}_r : \mathcal{G}^* \rightarrow \mathcal{G}$ by the rule

$$\langle u, \mathcal{O}(v) \rangle = \langle u \otimes v, r \rangle, \quad \forall u, v \in \mathcal{G}^*. \quad (2.1)$$

Conversely, this equality attaches an element $r \in \mathcal{G}^{\otimes 2}$ to every operator $\mathcal{O} : \mathcal{G}^* \rightarrow \mathcal{G}$. (Why do such banalities deserve being mentioned? Because they are not true in general. Please bear with me). The skewsymmetry of r is equivalent to skewsymmetry of \mathcal{O} :

$$\langle u, \mathcal{O}(v) \rangle + \langle v, \mathcal{O}(u) \rangle = 0. \quad (2.2)$$

Now, suppose temporarily that r is *nondegenerate*, i.e., \mathcal{O} is invertible. (\mathcal{G} is then even-dimensional). Consider the skewsymmetric bilinear form $\omega = \omega_r$ on \mathcal{G} :

$$\omega(x, y) = \langle \mathcal{O}^{-1}(x), y \rangle, \quad \forall x, y \in \mathcal{G}. \quad (2.3)$$

Theorem 2.4 (Drinfel'd [2]). *A nondegenerate r in $\wedge^2 \mathcal{G}$ satisfies CYBE iff ω is a 2-cocycle on \mathcal{G} .*

We postpone the proof of this Theorem until later on in this Section, since we aim at a higher prize: to reformulate this 2-cocycle characterization of classical r -matrices into a form suitable for a fruitful definition.

Let's write down the condition for ω to be a 2-cocycle on \mathcal{G} :

$$0 = \omega(x, [y, z]) + \text{c.p.} = \langle \mathcal{O}^{-1}(x), [y, z] \rangle + \text{c.p.}, \quad \forall x, y, z \in \mathcal{G}. \quad (2.4)$$

Since \mathcal{O} is invertible, we can find $u, v, w \in \mathcal{G}^*$ such that

$$x = \mathcal{O}(u), \quad y = \mathcal{O}(v), \quad z = \mathcal{O}(w). \quad (2.5)$$

The 2-cocycle condition (2.4) then becomes

$$\langle u, [\mathcal{O}(v), \mathcal{O}(w)] \rangle + \text{c.p.} = 0, \quad \forall u, v, w \in \mathcal{G}^*. \quad (2.6)$$

What have we achieved? First, in equality (2.6) the map \mathcal{O} is no longer required to be invertible. (2-cocycles are often degenerate. This is true in Fluid Mechanics and Plasma Physics, see many examples in [6]; same happens in finite dimensions, e.g. for complex semisimple Lie algebras, see discussion in [1], p. 62.) Second, the equality (2.6) is trilinear in \mathcal{G}^* , but we can transform this trilinear equation into an equivalent bilinear one, as follows.

Denote the coadjoint action of $x \in \mathcal{G}$ on $u \in \mathcal{G}^*$ by $x \cdot u$:

$$\langle x \cdot u, y \rangle = -\langle u, [x, y] \rangle, \quad \forall y \in \mathcal{G}. \quad (2.7)$$

Now, using the skewsymmetry of \mathcal{O} , we get

$$1) \quad \langle u, [\mathcal{O}(v), \mathcal{O}(w)] \rangle = -\langle \mathcal{O}(v) \cdot u, \mathcal{O}(w) \rangle = \langle w, \mathcal{O}(\mathcal{O}(v) \cdot u) \rangle, \quad (2.8a)$$

$$2) \quad \langle v, [\mathcal{O}(w), \mathcal{O}(u)] \rangle = \langle \mathcal{O}(u) \cdot v, \mathcal{O}(w) \rangle = -\langle w, \mathcal{O}(\mathcal{O}(u) \cdot v) \rangle. \quad (2.8b)$$

Substituting (2.8) into (2.6) we obtain

$$\langle w, \mathcal{O}(\mathcal{O}(v) \cdot u - \mathcal{O}(u) \cdot v) + [\mathcal{O}(u), \mathcal{O}(v)] \rangle = 0. \quad (2.9)$$

Since w is arbitrary, the 2-cocycle condition (2.6) is equivalent to the equality

$$\mathcal{O}(\mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u) = [\mathcal{O}(u), \mathcal{O}(v)], \quad \forall u, v \in \mathcal{G}^*. \quad (2.10)$$

This suggests the following generalization of the notion of the classical r -matrix. Let \mathcal{G} be a Lie algebra, \mathcal{U} a \mathcal{G} -module, and $\mathcal{O} : \mathcal{U} \rightarrow \mathcal{G}$ a linear map. Let's make \mathcal{U} into an algebra by defining a skew multiplication $[\cdot, \cdot]$ in \mathcal{U} by the rule

$$[u, v] = \mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u, \quad \forall u, v \in \mathcal{U}. \quad (2.11)$$

\mathcal{O} is called an \mathcal{O} -operator, or a classical r -matrix, iff \mathcal{O} is a homomorphism of algebras:

$$\mathcal{O}([u, v]) = [\mathcal{O}(u), \mathcal{O}(v)], \quad \forall u, v \in \mathcal{U}; \quad (2.12)$$

equivalently,

$$\mathcal{O}(\mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u) = [\mathcal{O}(u), \mathcal{O}(v)], \quad \forall u, v \in \mathcal{U}. \quad (2.13)$$

Example 2.14. Let $\mathcal{G} = sl_2$ with a basis $(h; e; f)$:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \quad (2.15)$$

Let \mathcal{U} be 2-dimensional, with a basis $(v_0; v_1)$ and the action of sl_2 of the fundamental representation:

$$\begin{aligned} e.v_0 &= 0, & h.v_0 &= v_0, & f.v_0 &= v_1, \\ e.v_1 &= v_0, & h.v_1 &= -v_1, & f.v_1 &= 0. \end{aligned} \quad (2.16)$$

Each of the following 2 maps can be easily seen to be an \mathcal{O} -operator:

$$\mathcal{O} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = c_1 \begin{pmatrix} h \\ f \end{pmatrix} + c_2 \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad c_1, c_2 \text{ are constants}, \quad (2.17a)$$

$$\mathcal{O} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = c_3 \begin{pmatrix} e \\ -h \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ e \end{pmatrix}, \quad c_3, c_4 \text{ are constants}, \quad (2.17b)$$

Proposition 2.18. *If $\mathcal{O} : \mathcal{U} \rightarrow \mathcal{G}$ is an \mathcal{O} -operator then \mathcal{U} is a Lie algebra.*

Proof. By formula (2.11),

$$\begin{aligned} [[u, v], w] + \text{c.p.} &= \{\mathcal{O}([u, v]).w - \mathcal{O}(w).[u, v]\} + \text{c.p.} \\ &\stackrel{[\text{by (2.12)}]}{=} \{[\mathcal{O}(u), \mathcal{O}(v)].w - \mathcal{O}(w).(\mathcal{O}(u).v - \mathcal{O}(v).u)\} + \text{c.p.} \\ &\stackrel{[\text{since } \mathcal{U} \text{ is a } \mathcal{G}\text{-module}]}{=} \{(\mathcal{O}(u).(\mathcal{O}(v).w) - \mathcal{O}(v).(\mathcal{O}(u).w)) + \text{c.p.}\} \\ &\quad - \{\mathcal{O}(w).(\mathcal{O}(u).v) + \text{c.p.}\} + \{\mathcal{O}(w).(\mathcal{O}(v).u) + \text{c.p.}\} \\ &= \{(\mathcal{O}(u).(\mathcal{O}(v).w) - \mathcal{O}(v).(\mathcal{O}(u).w)) + \text{c.p.}\} \\ &\quad - \{\mathcal{O}(u).(\mathcal{O}(v).w) + \text{c.p.}\} + \{\mathcal{O}(v).(\mathcal{O}(u).w) + \text{c.p.}\} = 0. \end{aligned} \quad \blacksquare$$

Thus, \mathcal{O} is a posteriori a homomorphism of Lie algebras. (This explains formulae (2.17), since \mathcal{U} is 2-dimensional and \mathcal{O} maps \mathcal{U} into b_+ or b_- .) We shall not wander into the general \mathcal{U} -route here. (For example, adding *linear* on \mathcal{O} conditions making $\mathcal{G} + \mathcal{U}$ into a Lie algebra.) From now on \mathcal{U} reverts to the old familiar \mathcal{G}^* .

Proposition 2.19. *Let $\mathcal{O} : \mathcal{G}^* \rightarrow \mathcal{G}$ be an \mathcal{O} -operator, so that \mathcal{G}^* is now a Lie algebra. Then the skewsymmetric bilinear form Ω on \mathcal{G}^* :*

$$\Omega(u, v) = \langle u, \mathcal{O}(v) \rangle \quad (2.20)$$

is a 2-cocycle on \mathcal{G}^ .*

Proof. We have:

$$\begin{aligned} \Omega([u, v], w) + \text{c.p.} &= \langle [u, v], \mathcal{O}(w) \rangle + \text{c.p.} = -\langle w, \mathcal{O}([u, v]) \rangle + \text{c.p.} \\ &\stackrel{[\text{by (2.12)}]}{=} -\langle w, [\mathcal{O}(u), \mathcal{O}(v)] \rangle + \text{c.p.} \stackrel{[\text{by (2.6)}]}{=} 0. \end{aligned} \quad \blacksquare$$

Let us prove now Drinfel'd's Theorem 2.4. We shall evaluate each of the 3 terms in the \mathcal{O} -equation (2.13) and compare them to the expressions (1.29) for $c(r)$.

First, for $r = \sum_i (a_i \otimes b_i - b_i \otimes a_i)$ (1.28), we get

$$\begin{aligned} \langle u \otimes v, r \rangle &= \sum_i (\langle u, a_i \rangle \langle v, b_i \rangle - \langle u, b_i \rangle \langle v, a_i \rangle) \\ &= \langle u, \sum_i (\langle v, b_i \rangle a_i - \langle v, a_i \rangle b_i) \rangle, \end{aligned}$$

so that

$$\mathcal{O}(v) = \sum_i (\langle v, b_i \rangle a_i - \langle v, a_i \rangle b_i). \quad (2.21)$$

Now,

$$\begin{aligned} 1) \quad \mathcal{O}(\mathcal{O}(u) \cdot v) &= \mathcal{O} \left(\sum_j (\langle u, b_j \rangle (a_j \cdot v) - \langle u, a_j \rangle (b_j \cdot v)) \right) \\ &= \sum_{ji} (\langle u, b_j \rangle \langle a_j \cdot v, b_i \rangle a_i - \langle u, a_j \rangle \langle b_j \cdot v, b_i \rangle a_i \\ &\quad - \langle u, b_j \rangle \langle a_j \cdot v, a_i \rangle b_i + \langle u, a_j \rangle \langle b_j \cdot v, a_i \rangle b_i) \\ &= \sum_{ij} (\langle u, b_j \rangle \langle v, [b_i, a_j] \rangle a_i - \langle u, a_j \rangle \langle v, [b_i, b_i] \rangle a_i \\ &\quad - \langle u, b_j \rangle \langle v, [a_i, a_j] \rangle b_i + \langle u, a_j \rangle \langle v, [a_i, b_j] \rangle b_i) \\ &\stackrel{[\text{by (1.29b)}]}{=} - (\langle u, \rangle \otimes \langle v, \rangle \otimes \mathbf{1})([r^{12}, r^{23}]); \end{aligned} \quad (2.22a)$$

Interchanging u and v in the above calculation, we get

$$\begin{aligned} 2) \quad -\mathcal{O}(\mathcal{O}(v) \cdot u) &= \sum_{ij} (\langle u, [a_j, b_i] \rangle \langle v, b_j \rangle a_i + \langle u, [b_i, b_j] \rangle \langle v, a_j \rangle a_i \\ &\quad + \langle u, [a_i, a_j] \rangle \langle v, b_j \rangle b_i - \langle u, [a_i, b_j] \rangle \langle v, a_j \rangle b_i) \\ &\stackrel{[\text{by (1.29a)}]}{=} - (\langle u, \rangle \otimes \langle v, \rangle \otimes \mathbf{1})([r^{12}, r^{13}]); \end{aligned} \quad (2.22b)$$

$$\begin{aligned} 3) \quad [\mathcal{O}(v), \mathcal{O}(u)] &= \sum_{ij} [\langle v, b_i \rangle a_i - \langle v, a_i \rangle b_i, \langle u, b_j \rangle a_j - \langle u, a_j \rangle b_j] \\ &= \sum_{ij} (\langle u, b_j \rangle \langle v, b_i \rangle [a_i, a_j] - \langle u, a_j \rangle \langle v, b_i \rangle [a_i, b_j] \\ &\quad - \langle u, b_j \rangle \langle v, a_i \rangle [b_i, a_j] + \langle u, a_j \rangle \langle v, a_i \rangle [b_i, b_j]) \\ &\stackrel{[\text{by (1.29c)}]}{=} - (\langle u, \rangle \otimes \langle v, \rangle \otimes \mathbf{1})([r^{13}, r^{23}]). \end{aligned} \quad (2.22c)$$

Altogether, we thus find

$$\langle w, [\mathcal{O}(u), \mathcal{O}(v)] - \mathcal{O}(\mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u) \rangle = \langle u \otimes v \otimes w, c(r) \rangle, \quad \forall u, v, w \in \mathcal{G}^*. \quad (2.23)$$

3 Differential Lie Algebras say: 2-cocycles, – Si, r -matrices, – No, \mathcal{O} -operators are welcome

Let R be a commutative ring or algebra, and $\partial_1, \dots, \partial_m, : R \rightarrow R$ be m commuting derivations. A Lie algebra over R is R^N , some $N \in \mathbf{N}$, with a skew multiplication $[\cdot, \cdot] : R^N \times R^N \rightarrow R^N$ given by bilinear *differential operators*, such that

$$[[x, y], z] + \text{c.p.} = 0, \quad \forall x, y, z \in \tilde{R}^N, \quad (3.1)$$

where $\tilde{R} \supset R$ is *arbitrary* differential extension of R . (This means that the skewsymmetry of $[\cdot, \cdot]$ and Jacobi identity are the properties solely of differential operators performing the multiplication $[\cdot, \cdot]$ and are not dependent upon the quirks of R itself).

In this Section we consider the very simplest case $m = 1$. Denote ∂_1 by ∂ . First, let \mathcal{D}_1 be the Lie algebra of vector fields on the line: $\mathcal{D}_1 = R$ and

$$[X, Y] = XY' - X'Y, \quad \forall X, Y \in \mathcal{D}_1, \quad (3.2)$$

where $(\cdot)' := \partial(\cdot)$.

Consider the following bilinear form on \mathcal{D}_1 :

$$\omega(X, Y) = \partial^3(X)Y. \quad (3.3)$$

This form is skewsymmetric, in *differential sense*, for

$$\omega(X, Y) \sim -\omega(Y, X), \quad \forall X, Y \in \mathcal{D}_1, \quad (3.4)$$

where $a \sim b$ means that $(a - b) \in \text{Im } \partial$. Also,

$$\omega(X, [Y, Z]) + \text{c.p.} \sim 0, \quad \forall X, Y, Z \in \mathcal{D}_1, \quad (3.5)$$

so that ω is a (generalized) 2-cocycle. (All the necessary details of the theory can be found in [6].) But if we try to represent this 2-cocycle ω in the \mathcal{O} -form (2.3),

$$\omega(X, Y) = \langle \mathcal{O}^{-1}(X), Y \rangle, \quad (3.6)$$

we find that $\mathcal{O} = \partial^{-3}$: in other words, \mathcal{O} doesn't exist, and neither does r . We circumvent this particular obstacle as follows.

Denote by $\mathcal{G}(\mu)$, $\mu = \text{const} \in \mathcal{F} := \text{Ker } \partial|_R$, the following Lie algebra structure on $R + R$:

$$\left[\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right] = \begin{pmatrix} XY' - X'Y \\ (Xg - Yf + \mu(X'Y'' - X''Y'))' \end{pmatrix}. \quad (3.7)$$

We still have the 2-cocycle ω (3.3) on $\mathcal{G}(\mu)$:

$$\omega \left(\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right) = \partial^3(X)Y, \quad (3.8)$$

and it is still degenerate. However, $\mathcal{G}(\mu)$ also possess a *nondegenerate* symplectic 2-cocycle.

$$\Omega \left(\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right) = Xg - Yf. \quad (3.9)$$

Indeed,

$$\begin{aligned}
& \Omega \left(\begin{pmatrix} X \\ f \end{pmatrix}, \left[\begin{pmatrix} Y \\ g \end{pmatrix}, \begin{pmatrix} Z \\ h \end{pmatrix} \right] \right) + \text{c.p.} \\
&= \Omega \left(\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} YZ' - Y'Z \\ (Yh - Zg + \mu(Y'Z'' - Y''Z'))' \end{pmatrix} \right) + \text{c.p.} \\
&= \{ -(YZ' - Y'Z)f + X(Y'h + Yh' - Z'g - Zg' + \mu Y'Z''' - \mu Y'''Z') \} + \text{c.p.} \\
&= \mu X(Y'Z''' - Y'''Z') + \text{c.p.} \sim -\mu X'(Y'Z''' - Y'''Z') + \text{c.p.} = 0.
\end{aligned}$$

Taking the sum $\epsilon\omega + \Omega$ as the new 2-cocycle on $\mathcal{G}(\mu)$, $\epsilon = \text{const}$, we get

$$\begin{aligned}
& (\epsilon\omega + \Omega) \left(\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right) \\
&= \epsilon\partial^3(X)Y + (Xg - Yf) = \left\langle \begin{pmatrix} \epsilon\partial^3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right\rangle,
\end{aligned} \tag{3.10}$$

so that

$$\mathcal{O}^{-1} = \begin{pmatrix} \epsilon\partial^3 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.11}$$

and thus

$$\mathcal{O} = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon\partial^3 \end{pmatrix}. \tag{3.12}$$

Since \mathcal{O} is a *differential operator*, it corresponds to no element of $\wedge^2\mathcal{G}(\mu)$; it is only in finite dimensions that we can identify \mathcal{G} with $\text{Hom}(\mathcal{G}^*, R)$ and $\mathcal{G}^{\otimes 2}$ with $\text{Hom}(\mathcal{G}^{*\otimes 2}, R)$. The conclusion is inescapable: the proper notion of the classical r -matrix is a skewsymmetric \mathcal{O} -operator $\mathcal{G}^* \rightarrow \mathcal{G}$ satisfying the classical \mathcal{O} -defining equation (2.10).

In finite dimensions, there exists a different version of the notion of classical r -matrix, due to Semeynov-Tyan-Shansky [13]. It is already in operator form, acting as $r : \mathcal{G} \rightarrow \mathcal{G}$; but it requires \mathcal{G} to have an invariant nondegenerate scalar product, a condition rarely encountered in differential situations. [E.g., the Lie algebra $\mathcal{G}(\mu)$ (3.7) has no invariant scalar product, no matter what μ is.)

We conclude this Section by calculating the Lie algebra structure on $\mathcal{G}(\mu)^*$ induced by the \mathcal{O} -operator (3.12) via formula (2.11).

Denote typical elements in $\mathcal{G}(\mu)^*$ as $\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}$, with the pairing

$$\left\langle \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} X \\ f \end{pmatrix} \right\rangle = uX + pf. \tag{3.13}$$

Let us first obtain the formula for the coadjoint action of $\mathcal{G}(\mu)$ on $\mathcal{G}(\mu)^*$:

$$\begin{aligned}
& \left\langle \begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right\rangle \sim - \left\langle \begin{pmatrix} u \\ p \end{pmatrix}, \left[\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right] \right\rangle \\
& \stackrel{[\text{by (3.7)}]}{\sim} -u(XY' - X'Y) + p'(Xg - Yf + \mu X'Y'' - \mu X''Y') \\
& \sim ((uX)' + uX')Y - fp'Y + Xp'g + ((\mu X'p')'' + (\mu X''p')')Y.
\end{aligned}$$

Thus,

$$\begin{pmatrix} X \\ f \end{pmatrix} \cdot \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} X\partial + 2X' & (\mu\partial(X'\partial + 2X'') - f)\partial \\ 0 & X\partial \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}. \quad (3.14)$$

Hence,

$$\begin{aligned} \mathcal{O} \begin{pmatrix} u \\ p \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} &= \begin{pmatrix} p \\ -u + \epsilon p''' \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} \\ &= \begin{pmatrix} (pv' + 2p'v) + \mu((p'q')'' + (p''q')') - (-u + \epsilon p''')q' \\ pq' \end{pmatrix}. \end{aligned} \quad (3.15)$$

Therefore,

$$\begin{aligned} \left[\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right] &= \mathcal{O} \begin{pmatrix} u \\ p \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} - \mathcal{O} \begin{pmatrix} v \\ q \end{pmatrix} \cdot \begin{pmatrix} u \\ p \end{pmatrix} \\ &= \begin{pmatrix} (pv - qu + (\epsilon - \mu)(p'q'' - p''q'))' \\ pq' - p'q \end{pmatrix}. \end{aligned} \quad (3.16)$$

We see that

$$\mathcal{G}(\mu)^* \approx \mathcal{G}(\epsilon - \mu). \quad (3.17)$$

Since $\mathcal{G}(0) = \mathcal{D}_1 \ltimes V_1$ is certainly a Lie algebra, $\mathcal{G}(\mu)$ is so a posteriori:

$$\mathcal{G}(\epsilon) \approx \mathcal{G}(0)^*. \quad (3.18)$$

The reader who hasn't bothered to check the Jacobi identity for the Lie bracket (3.7) on $\mathcal{G}(\mu)$ may now feel smug about it. The reader who didn't blink an eye when the symplectic 2-cocycle (3.9) was sprung out on him as a *deus ex machina* without an explanation, as this is how modern mathematics is supposed to operate, will be disappointed to find a general construction of symplectic r -matrices in Appendix A2. Sorry about that.

4 \mathcal{O} -natural property of the \mathcal{O} -operators

Let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ be a homomorphism of Lie algebras. If everything is finite-dimensional and $r \in \wedge^2 \mathcal{G}$ then $\varphi(r) \in \wedge^2 \mathcal{H}$, and

$$c(\varphi(r)) = \varphi(c(r)); \quad (4.1)$$

thus, if r is a classical r -matrix then so is $\varphi(r)$.

Consider now the general case. Let $\mathcal{O} = \mathcal{O}_{\mathcal{G}} : \mathcal{G}^* \rightarrow \mathcal{G}$ be an \mathcal{O} -operator. Recall that this means that

$$\mathcal{O}(\mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u) = [\mathcal{O}(u), \mathcal{O}(v)], \quad \forall u, v \in \mathcal{G}^*, \quad (4.2)$$

and that \mathcal{O} is skewsymmetric:

$$\langle u, \mathcal{O}(v) \rangle + \langle v, \mathcal{O}(u) \rangle \sim 0, \quad \forall u, v \in \mathcal{G}^*. \quad (4.3)$$

Since

$$\langle u, \mathcal{O}(v) \rangle + \langle v, \mathcal{O}(u) \rangle \sim \mathbf{u}^t(\mathcal{O} + \mathcal{O}^\dagger)(\mathbf{v}), \quad (4.4)$$

\mathcal{O} is skewsymmetric iff

$$\mathcal{O}^\dagger = -\mathcal{O}, \quad (4.5)$$

where \mathcal{O}^\dagger is the operator adjoint to \mathcal{O} , and \mathbf{u} and \mathbf{v} are treated as column-vectors (see [6].)

Now let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ be a homomorphism of Lie algebras. It induces the dual map $\varphi^* : \mathcal{H}^* \rightarrow \mathcal{G}^*$. Since

$$\langle \varphi^*(\bar{u}), x \rangle = \langle \bar{u}, \varphi(x) \rangle, \quad \forall \bar{u} \in \mathcal{H}^*, \quad \forall x \in \mathcal{G}, \quad (4.6)$$

and

$$\langle \bar{u}, \varphi(x) \rangle = \bar{\mathbf{u}}^t \varphi(x) \sim (\varphi^\dagger(\bar{\mathbf{u}}))^t x, \quad (4.7)$$

we see that

$$\varphi^* = \varphi^\dagger. \quad (4.8)$$

Proposition 4.9. *Set*

$$\mathcal{O}_{\mathcal{H}} = \varphi \mathcal{O}_{\mathcal{G}} \varphi^* : \mathcal{H}^* \rightarrow \mathcal{H}. \quad (4.10)$$

Then $\mathcal{O}_{\mathcal{H}}$ is an \mathcal{O} -operator.

Proof. First, let's check that $\mathcal{O}_{\mathcal{H}}$ is skewsymmetric. We have:

$$(\mathcal{O}_{\mathcal{H}})^\dagger = (\varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger)^\dagger = \varphi(\mathcal{O}_{\mathcal{G}})^\dagger \varphi^\dagger = -\varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger = -\mathcal{O}_{\mathcal{H}}. \quad (4.11)$$

Next, for any $\bar{u}, \bar{v} \in \mathcal{H}^*$, we have to verify that

$$\mathcal{O}_{\mathcal{H}}(\mathcal{O}_{\mathcal{H}}(\bar{u}) \cdot \bar{v} - \mathcal{O}_{\mathcal{H}}(\bar{v}) \cdot \bar{u}) = [\mathcal{O}_{\mathcal{H}}(\bar{u}), \mathcal{O}_{\mathcal{H}}(\bar{v})]. \quad (4.12)$$

For the LHS of the expression (4.12) we get

$$\varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger (\varphi \mathcal{O}_{\mathcal{G}}^\dagger(\bar{u}) \cdot \bar{v} - \varphi \mathcal{O}_{\mathcal{G}}^\dagger(\bar{v}) \cdot \bar{u}), \quad (4.13L)$$

and for the RHS of the expression (4.12) we obtain

$$\begin{aligned} & [\varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger(\bar{u}), \varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger(\bar{v})] \stackrel{[\text{since } \varphi \text{ is a homomorphism}]}{=} \varphi [\mathcal{O}_{\mathcal{G}} \varphi^\dagger(\bar{u}), \mathcal{O}_{\mathcal{G}} \varphi^\dagger(\bar{v})] \\ & \stackrel{[\text{since } \mathcal{O}_{\mathcal{G}} \text{ is an } \mathcal{O}\text{-operator}]}{=} \varphi \mathcal{O}_{\mathcal{G}} (\mathcal{O}_{\mathcal{G}} \varphi^\dagger(\bar{u}) \cdot \varphi^\dagger(\bar{v}) - \mathcal{O}_{\mathcal{G}} \varphi^\dagger(\bar{v}) \cdot \varphi^\dagger(\bar{u})), \end{aligned} \quad (4.13R)$$

and by formula (4.15) below the expressions (4.13L, R) are equal. ■

Lemma 4.14.

$$\varphi^\dagger(\varphi(X) \cdot \bar{v}) = X \cdot \varphi^\dagger(\bar{v}), \quad \forall X \in \mathcal{G}, \quad \forall \bar{v} \in \mathcal{H}^*. \quad (4.15)$$

Proof. Formula (4.15) is an equality in \mathcal{G}^* . Any such equality, $(\cdot) = (\cdot)$, is equivalent to the relation

$$\langle (\cdot), Y \rangle \sim \langle (\cdot), Y \rangle, \quad \forall Y \in \mathcal{G}. \quad (4.16)$$

So,

$$\begin{aligned} \langle \varphi^\dagger(\varphi(X) \cdot \bar{v}), Y \rangle &\sim \langle \varphi(X) \cdot \bar{v}, \varphi(Y) \rangle \sim -\langle \bar{v}, [\varphi(X), \varphi(Y)] \rangle \\ &\stackrel{[\text{since } \varphi \text{ is a homomorphism}]}{=} -\langle \bar{v}, \varphi([X, Y]) \rangle \sim -\langle \varphi^\dagger(\bar{v}), [X, Y] \rangle \sim \langle X \cdot \varphi^\dagger(\bar{v}), Y \rangle. \quad \blacksquare \end{aligned}$$

In § 6 we establish that a quadratic Poisson bracket on \mathcal{G}^* canonically associated to every \mathcal{O} -operator $\mathcal{O}_{\mathcal{G}}$, also has the natural property.

Remark 4.17. The map $\varphi^* : \mathcal{H}^* \rightarrow \mathcal{G}^*$ is a homomorphism of Lie algebras.

Proof. Take any $\bar{u}, \bar{v} \in \mathcal{H}^*$. We have to show that

$$\varphi^\dagger([\bar{u}, \bar{v}]) = [\varphi^\dagger(\bar{u}), \varphi^\dagger(\bar{v})], \quad (4.18)$$

which is

$$\varphi^\dagger(\mathcal{O}_{\mathcal{H}}(\bar{u}) \cdot \bar{v} - \mathcal{O}_{\mathcal{H}}(\bar{v}) \cdot \bar{u}) = \mathcal{O}_{\mathcal{G}}\varphi^\dagger(\bar{u}) \cdot \varphi^\dagger(\bar{v}) - \mathcal{O}_{\mathcal{G}}\varphi^\dagger(\bar{v}) \cdot \varphi^\dagger(\bar{u}),$$

which further is

$$\varphi^\dagger(\varphi \mathcal{O}_{\mathcal{G}}\varphi^\dagger(\bar{v}) \cdot \bar{v} - \varphi \mathcal{O}_{\mathcal{G}}\varphi^\dagger(\bar{v}) \cdot \bar{u}) = \mathcal{O}_{\mathcal{G}}\varphi^\dagger(\bar{u}) \cdot \varphi^\dagger(\bar{v}) - \mathcal{O}_{\mathcal{G}}\varphi^\dagger(\bar{v}) \cdot \varphi^\dagger(\bar{u}),$$

and this is true by formula (4.15). ■

Remark 4.19. In the generality we are working, many finite-dimensional notions disappear. For example, Hamilton–Lie groups and their Hamiltonian actions, – though *infinitesimal* versions of those may survive, see § 5, 6. Other things disappear altogether, such as representation of commutator on \mathcal{G}^* by the cocommutator $\mathcal{G} \rightarrow \mathcal{G} \otimes \mathcal{G}$. Every time one needs to use the finite-dimensional isomorphism $\text{End}(V) \approx V^* \otimes V$, one gets into all sorts of trouble with the authorities, since $\text{Diff}(V)$ is infinite-dimensional no matter what dimension of V is.

5 Linear Poisson brackets on dual spaces to Lie algebras

Before we tackle *quadratic* Poisson brackets on \mathcal{G}^* , it's instructive to review the *linear* Poisson brackets; this way we can introduce basic definitions and themes in more familiar surroundings.

So, let $\mathcal{G} = R^N$ be a differential Lie algebra. (Or differential-difference one; it's almost the same, as far as the theory goes, so I prefer not to clutter the presentation with indices corresponding to discrete degrees of freedom. See Remark 5.50 below for more details.) The dual space \mathcal{G}^* is also R^N . The differential ring $C = C_u = R[u_i^{(\sigma)}]$, $i = 1, \dots, N$, $\sigma \in \mathbf{Z}_+^m$, is what used to be the ring $\text{Fun}(\mathcal{G}^*)$ of smooth functions on \mathcal{G}^* in finite dimensions.

On the ring C_u we have the Poisson bracket

$$\{H, F\} = X_H(F) \sim \frac{\delta F}{\delta \mathbf{u}^t} B \left(\frac{\delta H}{\delta \mathbf{u}} \right), \quad (5.1)$$

where the Hamiltonian matrix B , *linear* in u , is extracted from the following defining relation:

$$\{\mathbf{u}^t \mathbf{X}, \mathbf{u}^t \mathbf{Y}\} \sim \mathbf{u}^t [\mathbf{X}, \mathbf{Y}], \quad \forall \mathbf{X}, \mathbf{Y} \in \mathcal{G}. \quad (5.2)$$

This is the differential version of the more familiar form

$$\{\langle u, X \rangle, \langle u, Y \rangle\} \sim \langle u, [X, Y] \rangle, \quad \forall X, Y \in \mathcal{G}. \quad (5.3)$$

Since

$$\{u^t X, u^t Y\} \sim Y^t B(X), \quad (5.4a)$$

and

$$u^t [X, Y] = \langle u, [X, Y] \rangle \sim -\langle X^\cdot u, Y \rangle, \quad (5.4b)$$

we see that

$$B(X) = -X^\cdot u. \quad (5.5)$$

Thus, the linear Poisson bracket on \mathcal{G}^* has the form

$$\{H, F\} \sim -\left\langle \frac{\delta H}{\delta u}^\cdot u, \frac{\delta F}{\delta u} \right\rangle \quad (5.6a)$$

$$\sim \left\langle u, \left[\frac{\delta H}{\delta u}, \frac{\delta F}{\delta u} \right] \right\rangle. \quad (5.6b)$$

The *Casimirs* of a Poisson bracket are those Hamiltonians H for which the vector field $X_H = \{H, \cdot\}$ is identically zero. (In finite dimensions, the common level surfaces of Casimirs are symplectic leaves.) From formula (5.6a) we see that the Casimirs on \mathcal{G}^* , also called (for a reason) coadjoint invariants, are the solutions of the equation

$$\frac{\delta H}{\delta u}^\cdot u = 0. \quad (5.7)$$

Equivalently,

$$\langle X^\cdot u, \frac{\delta H}{\delta u} \rangle \sim 0, \quad \forall X \in \mathcal{G}. \quad (5.8)$$

Example 5.9. Let \mathcal{G} be the Lie algebra \mathcal{D}_1 of § 3:

$$[X, Y] = X\partial(Y) - \partial(X)Y, \quad \forall X, Y \in \mathcal{D}_1. \quad (5.10)$$

Then

$$u[X, Y] = u(XY' - X'Y) \sim -Y(u\partial + \partial u)(X). \quad (5.11)$$

Thus,

$$B = B(\mathcal{D}_1) = -(u\partial + \partial u) \quad (5.12)$$

$$= -2\sqrt{u}\partial\sqrt{u}. \quad (5.13)$$

Therefore,

$$H \in \text{Ker}(B) \Leftrightarrow \frac{\delta H}{\delta u} = \text{const}/\sqrt{u} \Leftrightarrow H = \text{const}\sqrt{u}. \quad (5.14)$$

Equivalently, from formula (3.14) we see that

$$X \cdot u = (X\partial + 2X')(u) = Xu' + 2X'u = \frac{1}{X}(X^2u)', \quad (5.15)$$

so that H is Casimir iff

$$\left(\frac{\delta H}{\delta u}\right)^2 u = \text{const} \Leftrightarrow H = \text{const}\sqrt{u}. \quad (5.16)$$

We see that in this case H belongs not to the ring C_u itself but to its algebraic extension.

Remark 5.17. In finite dimensions, the linear Poisson bracket (5.6) was discovered by Lie and rediscovered by everyone else.

Let us check that the linear Poisson bracket (5.6) is *natural*. Let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ be a homomorphism of Lie algebras. Let $C_q = R[q_j^{(\sigma)}]$ be the differential ring of functions on \mathcal{H}^* , $j = 1, \dots, \dim(\mathcal{H})$. Let $\Phi : C_u \rightarrow C_q$ be the differential homomorphism dual to the map of spaces $\varphi^* : \mathcal{H}^* \rightarrow \mathcal{G}^*$. To calculate Φ , it's enough to notice that C_u and C_q are (differentially) generated by linear functions on \mathcal{G}^* and \mathcal{H}^* respectively:

$$\Phi : \langle \cdot, X \rangle \mapsto \langle \cdot, \varphi(X) \rangle, \quad \forall X \in \mathcal{G}. \quad (5.18)$$

Thus,

$$\Phi(\langle u, X \rangle) = \Phi(u)^t X = \langle q, \varphi(X) \rangle = q^t \varphi(X) \sim \varphi^\dagger(q)^t X, \quad (5.19)$$

so that

$$\Phi(u) = \varphi^\dagger(q). \quad (5.20)$$

Denote by $B_{\mathcal{G}} = B(\mathcal{G})$ the linear Hamiltonian matrix associated to the Lie algebra \mathcal{G} by formula (5.5). The property of the matrix $B_{\mathcal{G}}$ being natural means that

$$\Phi(\{H, F\}_{\mathcal{G}^*}) = \{\Phi(H), \Phi(F)\}_{\mathcal{H}^*}, \quad \forall H, F \in C_{\mathcal{G}^*} = C_u. \quad (5.21)$$

By the well-known criterion (see, e.g. [6] p. 54), a map $\Phi : C_1 \rightarrow C_2$ is Hamiltonian between the Hamiltonian matrices B_1 and B_2 over rings C_1 and C_2 respectively, iff

$$\Phi(B_1) = D(\Phi)B_2D(\Phi)^\dagger, \quad (5.22)$$

where D stands for the Fréchet derivative and

$$\Phi = \Phi(q^1), \quad (5.23)$$

q^1 and q^2 being the column-vectors of generators of the rings C_1 and C_2 respectively. For the matrices $B_1 = B_{\mathcal{G}}$ and $B_2 = B_{\mathcal{H}}$, a proof of the identity (5.22) can be found in [6] p. 66. Instead of repeating this type of proof, – which becomes very cumbersome for the quadratic Poisson bracket on \mathcal{G}^* defined in § 6, – I will reformulate the criterion (5.22) into a very useful form:

Proposition 5.24. *A map $\Phi : C_1 \rightarrow C_2$ is Hamiltonian iff*

$$\Phi(\{H, F\}_1) \sim \{\Phi(H), \Phi(F)\}_2, \quad \forall H, F \text{ linear in } q^1. \quad (5.25)$$

Proof. We shall show that if H and F are *arbitrary linear* in \mathbf{q}^1 then formula (5.25) implies formula (5.22). Let

$$H = \mathbf{q}^{1t} \mathbf{X}, \quad F = \mathbf{q}^{1t} \mathbf{Y}, \quad \mathbf{X}, \mathbf{Y} \in \tilde{R}^{N_1},$$

so that

$$\{H, F\}_1 \sim \mathbf{Y}^t B_1(\mathbf{X}), \quad \Phi(\{H, F\}_1) \sim \mathbf{Y}^t \Phi(B_1)(\mathbf{X}), \quad (5.26)$$

$$\{\Phi(H), \Phi(F)\}_2 = \{\Phi^t \mathbf{X}, \Phi^t \mathbf{Y}\}_2 \sim \left(\frac{\delta}{\delta \mathbf{q}^2}(\Phi^t \mathbf{Y}) \right)^t B_2 \frac{\delta}{\delta \mathbf{q}^2}(\Phi^t \mathbf{X}). \quad (5.27)$$

Now, since \mathbf{X} and \mathbf{Y} are $q^{(\cdot)}$ -independent,

$$\frac{\delta}{\delta q_s}(\Phi^t \mathbf{X}) = D_{q_s}(\Phi)^\dagger(\mathbf{X}) = \sum_j D_{q_s}(\Phi_j)^\dagger(X_j), \quad (5.28)$$

so that

$$\begin{aligned} \left(\frac{\delta}{\delta \mathbf{q}^2}(\Phi^t \mathbf{Y}) \right)^t B_2 \frac{\delta}{\delta \mathbf{q}^2}(\Phi^t \mathbf{X}) &= \sum_{ij} [D_i(\Phi)^\dagger(\mathbf{Y})]^t (B_2)_{ij} D_j(\Phi)^\dagger(\mathbf{X}) \\ &\sim \sum_{ij} \mathbf{Y}^t D_i(\Phi)(B_2)_{ij} D_j(\Phi)^\dagger(\mathbf{X}) = \mathbf{Y}^t D(\Phi) B_2 D(\Phi)^\dagger(\mathbf{X}). \end{aligned} \quad (5.29)$$

Comparing formulae (5.26) and (5.29) and remembering that \mathbf{X} and \mathbf{Y} are arbitrary, we arrive at the Hamiltonian criterion (5.22). \blacksquare

Remark 5.30. A little more effort will show that the equality modulo $\sum_{\ell=1}^m \text{Im } \partial_\ell \text{ sign} \sim$ in formula (5.25) can be replaced by the exact equality $\text{sign} =$ (see [6] p. 53). We won't need this more precise form in what follows.

Assume for a moment that we are thrown back in time into finite dimensions. Let G be a Lie group whose Lie algebra is \mathcal{G} . Then G acts on \mathcal{G}^* by the coadjoint representation, and this action preserves the linear Poisson bracket on \mathcal{G}^* . This means that the map $\text{Ad}^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is Poisson, with the Poisson bracket on $G \times \mathcal{G}^*$ being the product of Poisson brackets on G and \mathcal{G}^* and Poisson bracket and G being zero. This is a particular case of the following more general set-up. Let G be a Hamilton–Lie group. (This is the original name given to the subject by its inventor, V.G. Drinfel'd [2]; subsequent commentators have changed the original name into “Poisson–Lie” groups). Let M be a Poisson manifold. Suppose G acts from the left on M in such a way that the action map $G \times M \rightarrow M$ is a Hamiltonian (= Poisson) map, with the Poisson structure on $G \times M$ being of product type. Then *infinitesimal* criterion for this action to be Hamiltonian is ([14])

$$\begin{aligned} X^\wedge(\{H, F\}) - \{X^\wedge(H), F\} - \{H, X^\wedge(F)\} &= \langle [\theta_H, \theta_F], X \rangle, \\ \forall H, F \in \text{Fun}(M), \quad \forall X \in \mathcal{G} = \text{Lie}(G). \end{aligned} \quad (5.31)$$

Here X^\wedge is the vector field $\left. \frac{d}{dt} \exp(tX)^* \right|_{t=0}$ on M generated by $X \in \mathcal{G}$, $\{, \}$ is the Poisson bracket on M , $\theta_H : M \rightarrow \mathcal{G}^*$, for a given function H on M , is the map defined by the rule

$$\theta_H(x) = d_g H(gx)|_{g=e}, \quad \forall x \in M, \quad (5.32)$$

and $[\theta_H, \theta_F]$ is the commutator in \mathcal{G}^* induced by the differential at the identity in G of the multiplicative Poisson bracket on G . (For a proof, see, e.g. [4] p. 45).

We aim to reformulate the infinitesimal criterion of Hamiltonian action (5.31) into a *definition* usable for the functional case where there are no Lie groups present anymore, only Lie algebras. Let \mathcal{G} be such Lie algebra, and let $^\wedge : \mathcal{G} \mapsto D^{ev}(C)$ be an antirepresentation of \mathcal{G} in the Lie algebra of evolution derivations of some differential ring C . Let $\sim : C \rightarrow C \otimes \mathcal{G}^*$ be the map defined by the relation

$$\langle H^\sim, X \rangle \sim X^\wedge(H), \quad \forall H \in C, \quad \forall X \in \mathcal{G}. \quad (5.33)$$

Extend the given commutator in \mathcal{G}^* , – whether given by an \mathcal{O} -operator, or otherwise, but making $\mathcal{G} + \mathcal{G}^*$ into a differential Lie algebra, see Appendix I, – into the one on $C \otimes \mathcal{G}^*$ by treating C as just another differential extension $\tilde{R} \supset R$. The criterion of infinitesimal Hamiltonian action then is:

$$\begin{aligned} X^\wedge(\{H, F\}) - \{X^\wedge(H), F\} - \{H, X^\wedge(F)\} &\sim \langle [H^\sim, F^\sim], X \rangle, \\ \forall H, F \in C, \quad \forall X \in \mathcal{G}, \end{aligned} \quad (5.34)$$

with $\{, \}$ being a Poisson bracket on C defined by some Hamiltonian matrix.

We are interested in this paper in the case $C = C_{\mathcal{G}^*} = C_u$. In this case the action of the evolution vector field X^\wedge on C_u corresponding to an element $X \in \mathcal{G}$ is given by the formula

$$X^\wedge(u) = X^\cdot u. \quad (5.35)$$

For the *linear* bracket on \mathcal{G}^* , the RHS of the criterion (5.34) vanishes identically since \mathcal{G}^* is considered as an abelian Lie algebra. Thus, we have to verify that

$$X^\wedge(\{H, F\}) \sim \{X^\wedge(H), F\} + \{H, X^\wedge(F)\}, \quad \forall H, F \in C_u. \quad (5.36)$$

However, this relation follows at once from the fact that X^\wedge is a *Hamiltonian* vector field with the Hamiltonian

$$G = -\langle u, X \rangle. \quad (5.37)$$

Indeed, by formula (5.5),

$$X_G(u) = B \left(\frac{\delta G}{\delta u} \right) = -\frac{\delta G}{\delta u} u = X^\cdot u, \quad (5.38)$$

and this is formula (5.35)

We now prove “the main result of the infinitesimal Hamiltonian action”:

Theorem 5.39. *The infinitesimal Hamiltonian action criterion (5.34) for a given Poisson bracket on \mathcal{G}^* is enough to verify for Hamiltonians H and F linear in the u ’s.*

Proof. We are going to show that each side of the criterion (5.34) can be transformed into a form which is a bilinear *differential* operator acting on the vectors

$$Y = \frac{\delta H}{\delta u}, \quad Z = \frac{\delta F}{\delta u}. \quad (5.40)$$

First by formula (5.33),

$$\langle H^\sim, X \rangle \sim X^\wedge(H) \sim (X^\wedge(\mathbf{u}))^t \frac{\delta H}{\delta \mathbf{u}} = \langle X^\cdot u, \mathbf{Y} \rangle \sim \langle u, [Y, X] \rangle \sim -\langle Y^\cdot u, X \rangle,$$

so that

$$H^\sim = -\frac{\delta H^\cdot}{\delta \mathbf{u}} \mathbf{u}. \quad (5.41)$$

Similarly, $F^\sim = -\mathbf{Z}^\cdot u$, and the RHS of the criterion (5.34) is therefore indeed a bilinear differential operator w.r.t. \mathbf{Y} and \mathbf{Z} .

Next, let B be an unspecified Hamiltonian matrix over the ring $C_u = C_{\mathcal{G}^*}$, so that

$$\{H, F\} \sim \frac{\delta F}{\delta \mathbf{u}^t} B \left(\frac{\delta H}{\delta \mathbf{u}} \right) = \mathbf{Z}^t B(\mathbf{Y}). \quad (5.42)$$

Transforming separately each of the 3 terms on the LHS of the criterion (5.34), we get:

$$\begin{aligned} 1) \quad X^\wedge(\{H, F\}) &\sim X^\wedge(\mathbf{Z}^t B(\mathbf{Y})) \\ &= (X^\wedge(\mathbf{Z}))^t B(\mathbf{Y}) + \mathbf{Z}^t X^\wedge(B)(\mathbf{Y}) + \mathbf{Z}^t B(X^\wedge(\mathbf{Y})); \end{aligned} \quad (5.43a)$$

$$\begin{aligned} 2) \quad -\{X^\wedge(H), F\} &\sim -\mathbf{Z}^t B \frac{\delta}{\delta \mathbf{u}} (X^\wedge(H)) \\ &\stackrel{[\text{by formula (5.46) below}]}{=} -\mathbf{Z}^t B([Y, X] + X^\wedge(\mathbf{Y})); \end{aligned} \quad (5.43b)$$

$$\begin{aligned} 3) \quad -\{H, X^\wedge(F)\} &\sim \{X^\wedge(F), H\} \\ &\stackrel{[\text{by (5.43b)}]}{\sim} \mathbf{Y}^t B([\mathbf{Z}, X] + X^\wedge(\mathbf{Z})) \sim -([\mathbf{Z}, X] + X^\wedge(\mathbf{Z}))^t B(\mathbf{Y}). \end{aligned} \quad (5.43c)$$

Adding up the expressions (5.43), we obtain

$$\begin{aligned} &X^\wedge(\{H, F\}) - \{X^\wedge(H), F\} - \{H, X^\wedge(F)\} \\ &\sim \frac{\delta F}{\delta \mathbf{u}^t} X^\wedge(B) \left(\frac{\delta H}{\delta \mathbf{u}} \right) + \frac{\delta F}{\delta \mathbf{u}^t} B \left(\left[X, \frac{\delta H}{\delta \mathbf{u}} \right] \right) + \left[X, \frac{\delta F}{\delta \mathbf{u}} \right]^t B \left(\frac{\delta H}{\delta \mathbf{u}} \right). \quad \blacksquare \end{aligned} \quad (5.44)$$

Lemma 5.45.

$$\frac{\delta}{\delta \mathbf{u}} (X^\wedge(H)) = \left[\frac{\delta H}{\delta \mathbf{u}}, X \right] + X^\wedge \left(\frac{\delta H}{\delta \mathbf{u}} \right). \quad (5.46)$$

Proof. We have,

$$X^\wedge(H) \sim (X^\wedge(\mathbf{u}))^t \frac{\delta H}{\delta \mathbf{u}} = \langle X^\cdot u, \frac{\delta H}{\delta \mathbf{u}} \rangle \sim \langle u, \left[\frac{\delta H}{\delta \mathbf{u}}, X \right] \rangle. \quad (5.47)$$

Therefore, since X is u -independent,

$$\frac{\delta}{\delta \mathbf{u}} (X^\wedge(H)) = \left[\frac{\delta H}{\delta \mathbf{u}}, X \right] + D \left(\frac{\delta H}{\delta \mathbf{u}} \right)^\dagger (X^\cdot u). \quad (5.48)$$

Now, since

$$D\left(\frac{\delta H}{\delta \mathbf{u}}\right)^\dagger = D\left(\frac{\delta H}{\delta \mathbf{u}}\right) \quad (5.49)$$

(see [6]), the second summand on the RHS of the expression (5.48) can be transformed into

$$D\left(\frac{\delta H}{\delta \mathbf{u}}\right)(X^\wedge(\mathbf{u})) = X^\wedge\left(\frac{\delta H}{\delta \mathbf{u}}\right). \quad \blacksquare$$

Remark 5.50. Exactly where have we used the restriction that our objects, – Lie algebras, rings, etc., – are differential rather than differential-difference ones? The answer is nowhere except in the notation: in the general case,

$$a \sim b \Leftrightarrow (a - b) \in \sum_{\ell=1}^m \text{Im } \partial_\ell + \sum_{g \in G} \text{Im } (\hat{g} - \hat{e}), \quad (5.51)$$

$C_u = R[u_i^{(g|\sigma)}]$, $g \in G$, $\sigma \in \mathbf{Z}_+^m$, etc., where G is a discrete group whose elements index the discrete degrees of freedom, \hat{g} is the action of the element $g \in G$ on R , C_u , etc. The presence of discrete degrees of freedom is hidden in the notation $\frac{\delta}{\delta \mathbf{u}}$, $(\cdot)^\dagger$, $D(\cdot)$, etc. (The continuous reader may safely ignore this Remark).

We conclude this Section by considering *affine* Poisson brackets. These brackets have the corresponding Hamiltonian operators of the form

$$B = B^{\text{lin}} + b, \quad (5.52)$$

where B^{lin} is a Hamiltonian matrix operator *linearly* dependent upon \mathbf{u} , and b is \mathbf{u} -independent. Thus,

$$\{H, F\} \sim \frac{\delta F}{\delta \mathbf{u}^t} B \left(\frac{\delta H}{\delta \mathbf{u}} \right) = \langle \mathbf{u}, \left[\frac{\delta H}{\delta \mathbf{u}}, \frac{\delta F}{\delta \mathbf{u}} \right] \rangle + \langle b \left(\frac{\delta H}{\delta \mathbf{u}} \right), \frac{\delta F}{\delta \mathbf{u}} \rangle, \quad (5.53)$$

where $[\cdot, \cdot]$ is a commutator in some Lie algebra, say \mathcal{G} , and $b : \mathcal{G} \rightarrow \mathcal{G}^*$ is a skewsymmetric operator defining a generalized 2-cocycle on \mathcal{G} :

$$\langle b([X, Y]), Z \rangle + \text{c.p.} \sim 0, \quad \forall X, Y, Z \in \mathcal{G} \quad (5.54)$$

(see [6].) Since this 2-cocycle is, in general, *generalized* (“ \sim ” instead of “ $=$ ” in the RHS of (5.54)), it does not correspond to a central extension any more. Nevertheless, we have

Theorem 5.55. *Let \mathcal{G} act on $C_{\mathcal{G}^*}$ by the rule*

$$X^\wedge(\mathbf{u}) = X \cdot \mathbf{u} - b(\mathbf{X}). \quad (5.56)$$

Then this action satisfies the infinitesimal Hamiltonian action criterion (5.34) for the affine Poisson bracket (5.53).

Proof. Let us write

$$X^\wedge = X_{\text{old}}^\wedge + X_{\text{new}}^\wedge, \quad \{H, F\} = \{H, F\}_{\text{old}} + \{H, F\}_{\text{new}}, \quad (5.57)$$

where

$$X_{\text{new}}^{\wedge}(\mathbf{u}) = -b(\mathbf{X}), \quad (5.58)$$

$$\{H, F\}_{\text{new}} = \left\langle b \left(\frac{\delta H}{\delta \mathbf{u}} \right), \frac{\delta F}{\delta \mathbf{u}} \right\rangle. \quad (5.59)$$

By Theorem 5.39, we need to verify the relation

$$X^{\wedge}(\{H, F\}) - \{X^{\wedge}(H), F\} - \{H, X^{\wedge}(F)\} \sim 0 \quad (5.60)$$

for all H, F linear in \mathbf{u} :

$$H = \langle u, Y \rangle, \quad F = \langle u, Z \rangle, \quad \forall Y, Z \in \mathcal{G}. \quad (5.61)$$

We have:

$$\begin{aligned} X^{\wedge}(\{H, F\}) &\sim (X_{\text{old}}^{\wedge} + X_{\text{new}}^{\wedge})\{\langle u, [Y, Z] \rangle + \langle b(Y), Z \rangle\} \\ &= X_{\text{old}}^{\wedge}(\{H, F\}_{\text{old}}) - \langle b(X), [Y, Z] \rangle, \end{aligned} \quad (5.62a)$$

$$\begin{aligned} -\{X^{\wedge}(H), F\} &= -\{\langle X^{\cdot} u - b(X), Y \rangle, \langle u, Z \rangle\} \\ &\sim -\{\langle u, [X, Y] \rangle, \langle u, Z \rangle\} = -\{X^{\text{old}}(H), F\}_{\text{old}} + \langle b([X, Y]), Z \rangle, \end{aligned} \quad (5.62b)$$

$$-\{H, X^{\wedge}(F)\} \sim -\{H, X_{\text{old}}^{\wedge}(F)\}_{\text{old}} - \langle b([X, Z]), Y \rangle. \quad (5.62c)$$

Adding the expressions (5.62) up and remembering formula (5.36), we obtain formula (5.60). ■

6 Quadratic Poisson brackets on dual spaces to Lie algebras

The action map $\text{Ad}^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is Poisson when the Poisson bracket on G is zero and on \mathcal{G}^* is linear. When G itself has a nonzero multiplicative Poisson bracket on it, *coming from* an r -matrix, there exists a quadratic deformation of the standard linear Poisson bracket on \mathcal{G}^* such that the action map $\text{Ad}^* : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ is still Poisson. (If the multiplicative Poisson bracket on G is *not* of r -matrix type, such quadratic deformation is, in general, impossible.) This was found by Kupersmidt and Stoyanov in [11]. In this Section we construct a differential (-difference) analog of this quadratic bracket, prove that it's compatible with the linear one, verify that this quadratic bracket is natural, and then check the infinitesimal Hamiltonian action criterion for it.

So, let $\mathcal{O} : \mathcal{G}^* \rightarrow \mathcal{G}$ be an \mathcal{O} -operator. Recall that \mathcal{O} is skewsymmetric:

$$\langle u, \mathcal{O}(v) \rangle \sim -\langle v, \mathcal{O}(u) \rangle, \quad \forall u, v \in \mathcal{G}^*, \quad (6.1)$$

and

$$\mathcal{O}(\mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u) = [\mathcal{O}(u), \mathcal{O}(v)], \quad \forall u, v \in \mathcal{G}^*. \quad (6.2)$$

Define the quadratic Poisson bracket on \mathcal{G}^* by formula

$$\{H, F\} \sim \left\langle \frac{\delta H}{\delta \mathbf{u}} \cdot \mathbf{u}, \mathcal{O} \left(\frac{\delta F}{\delta \mathbf{u}} \cdot \mathbf{u} \right) \right\rangle \sim - \left\langle \frac{\delta F}{\delta \mathbf{u}} \cdot \mathbf{u}, \mathcal{O} \left(\frac{\delta H}{\delta \mathbf{u}} \cdot \mathbf{u} \right) \right\rangle \quad (6.3a)$$

$$\sim \langle \mathcal{O} \left(\frac{\delta H}{\delta \mathbf{u}} \cdot \mathbf{u} \right) \cdot \mathbf{u}, \frac{\delta F}{\delta \mathbf{u}} \rangle. \quad (6.3b)$$

The corresponding Hamiltonian matrix B is therefore quadratic in u :

$$B(\mathbf{Y}) = \mathcal{O}(\mathbf{Y} \cdot \mathbf{u}) \cdot \mathbf{u}. \quad (6.4)$$

The quadratic bracket (6.3) is obviously skewsymmetric. Let us verify that it satisfies the Jacobi identity. By the main result of the Hamiltonian formalism ([6] p. 47), it is enough to check the Jacobi identity for Hamiltonians *linear* in \mathbf{u} . So, let

$$H = \mathbf{Y}^t \mathbf{u}, \quad F = \mathbf{Z}^t \mathbf{u}, \quad G = \mathbf{X}^t \mathbf{u}. \quad (6.5)$$

Then

$$\{H, F\} \sim \langle \mathbf{Y} \cdot \mathbf{u}, \mathcal{O}(\mathbf{Z} \cdot \mathbf{u}) \rangle \sim -\langle \mathbf{u}, [\mathbf{Y}, \mathcal{O}(\mathbf{Z} \cdot \mathbf{u})] \rangle \sim \langle \mathbf{u}, [\mathbf{Z}, \mathcal{O}(\mathbf{Y} \cdot \mathbf{u})] \rangle. \quad (6.6)$$

Therefore,

$$\frac{\delta \{H, F\}}{\delta \mathbf{u}} = -[\mathbf{Y}, \mathcal{O}(\mathbf{Z} \cdot \mathbf{u})] + [\mathbf{Z}, \mathcal{O}(\mathbf{Y} \cdot \mathbf{u})], \quad (6.7)$$

and hence, in the notation

$$\bar{X} = X \cdot u, \quad \bar{Y} = Y \cdot u, \quad \bar{Z} = Z \cdot u, \quad (6.8)$$

$$\begin{aligned} \{\{H, F\}, G\} + \text{c.p.} &\sim \langle -[Y, \mathcal{O}(\bar{Z})] \cdot u + [Z, \mathcal{O}(\bar{Y})] \cdot u, \mathcal{O}(\bar{X}) \rangle + \text{c.p.} \\ &= -(\langle [Y, \mathcal{O}(\bar{Z})] \cdot u, \mathcal{O}(\bar{X}) \rangle + \text{c.p.}) + (\langle [Y, \mathcal{O}(\bar{X})] \cdot u, \mathcal{O}(\bar{Z}) \rangle + \text{c.p.}) \\ &\sim \langle u, [[Y, \mathcal{O}(\bar{Z})], \mathcal{O}(\bar{X})] - [[Y, \mathcal{O}(\bar{X})], \mathcal{O}(\bar{Z})] \rangle + \text{c.p.} \\ &= \langle u, -[Y, [\mathcal{O}(\bar{X}), \mathcal{O}(\bar{Z})]] \rangle + \text{c.p.} \sim \langle \bar{Y}, [\mathcal{O}(\bar{X}), \mathcal{O}(\bar{Z})] \rangle + \text{c.p.} \end{aligned}$$

and this expression is ~ 0 by formula (2.6), itself equivalent to the \mathcal{O} -property (6.2).

Now let us verify that the linear and quadratic Poisson brackets on \mathcal{G}^* are compatible no matter what \mathcal{O} is. Recall that compatibility of two Poisson brackets means that their arbitrary linear combination with constant coefficients is again a Poisson bracket, i.e., it satisfies the Jacobi identity. This amounts to the relation

$$(\{\{H, F\}_1, G\}_2 + \{\{H, F\}_2, G\}_1) + \text{c.p.} \sim 0, \quad \forall H, G, F, \quad (6.9)$$

and the main Theorem of the Hamiltonian formalism asserts that this relation needs to be verified only for *linear* Hamiltonian H, F, G . So, for such H, F, G , given by formula (6.5), we have, by formulae (5.2) and (6.6)

$$\{H, F\}_1 \sim [\mathbf{Y}, \mathbf{Z}]^t \mathbf{u} \Rightarrow \{\{H, F\}_1, G\}_2 \sim \langle [\mathbf{Y}, \mathbf{Z}] \cdot \mathbf{u}, \mathcal{O}(\mathbf{X} \cdot \mathbf{u}) \rangle. \quad (6.10a)$$

On the other hand, by formulae (6.7) and (5.6), we get

$$\begin{aligned} \{\{H, F\}_2, G\}_1 &\sim \langle \mathbf{X} \cdot \mathbf{u}, -[\mathbf{Y}, \mathcal{O}(\mathbf{Z} \cdot \mathbf{u})] + [\mathbf{Z}, \mathcal{O}(\mathbf{Y} \cdot \mathbf{u})] \rangle \\ &\sim \langle \mathbf{Y} \cdot (\mathbf{X} \cdot \mathbf{u}), \mathcal{O}(\mathbf{Z} \cdot \mathbf{u}) \rangle - \langle \mathbf{Z} \cdot (\mathbf{X} \cdot \mathbf{u}), \mathcal{O}(\mathbf{Y} \cdot \mathbf{u}) \rangle. \end{aligned} \quad (6.10b)$$

Substituting expressions (6.10) into formula (6.9), we get

$$(\langle [\mathbf{X}, \mathbf{Y}] \cdot \mathbf{u} + \mathbf{Y}(\mathbf{X} \cdot \mathbf{u}) - \mathbf{X}(\mathbf{Y} \cdot \mathbf{u}), \mathcal{O}(\mathbf{Z} \cdot \mathbf{u}) \rangle) + \text{c.p.} = 0,$$

since

$$[\mathbf{X}, \mathbf{Y}] \cdot \mathbf{u} = \mathbf{X} \cdot (\mathbf{Y} \cdot \mathbf{u}) - \mathbf{Y} \cdot (\mathbf{X} \cdot \mathbf{u}). \quad (6.11)$$

If it so happens that the \mathcal{O} -operator $\mathcal{O} : \mathcal{G}^* \rightarrow \mathcal{G}$ is invertible, like in § 2, then we have a generalized 2-cocycle on \mathcal{G} :

$$\omega_b(X, Y) = \langle b(X), Y \rangle, \quad b = \epsilon \mathcal{O}^{-1}, \quad \epsilon = \text{const}, \quad (6.11a)$$

and thus a constant-coefficient Poisson bracket on $C_{\mathcal{G}^*}$:

$$\{H, F\}_0 \sim \langle b \left(\frac{\delta H}{\delta u} \right), \frac{\delta F}{\delta u} \rangle. \quad (6.11b)$$

Since ω_b is a generalized 2-cocycle on \mathcal{G} , this constant-coefficient Poisson bracket $\{ , \}_0$ on \mathcal{G}^* is compatible with the linear Poisson bracket $\{ , \}_1$. Let us verify that all three Poisson brackets on \mathcal{G}^* , – constant-coefficient, linear and quadratic, – are compatible. It remains only to verify compatibility of constant-coefficient $\{ , \}_0$ one and the quadratic $\{ , \}_2$ one. Again, for linear Hamiltonians (6.5), the Poisson bracket $\{H, F\}_0$ (6.11b) is u -independent, so that $\{\{H, F\}_0, G\}_2 = 0$. Thus, we need only to verify that

$$\{\{H, F\}_2, G\}_0 + \text{c.p.} \sim 0.$$

By formulae (6.7), (6.8), and (6.11), we have:

$$\begin{aligned} \{\{H, F\}_2, G\}_0 + \text{c.p.} &= \langle b \left(\frac{\delta \{H, F\}_2}{\delta u} \right), X \rangle + \text{c.p.} \\ &\sim \langle b(X), [Y, \mathcal{O}(\bar{Z})] - [Z, \mathcal{O}(\bar{Y})] \rangle + \text{c.p.} \\ &= (\langle b(X), [Y, \mathcal{O}(\bar{Z})] \rangle - \langle b(Y), [X, \mathcal{O}(\bar{Z})] \rangle) + \text{c.p.} \\ &\sim \langle -Y \cdot b(X) + X \cdot b(Y), \mathcal{O}(\bar{Z}) \rangle + \text{c.p.} \\ &\stackrel{[\text{by (2.11)}]}{=} \langle \epsilon[\mathcal{O}^{-1}(X), \mathcal{O}^{-1}(Y)], \mathcal{O}(\bar{Z}) \rangle + \text{c.p.} \\ &\sim -\epsilon \langle Z \cdot u, \mathcal{O}[\mathcal{O}^{-1}(X), \mathcal{O}^{-1}(Y)] \rangle + \text{c.p.} \\ &\stackrel{[\text{by (2.12)}]}{=} -\epsilon \langle Z \cdot u, [X, Y] \rangle + \text{c.p.} \sim \epsilon \langle u, [Z, [X, Y]] \rangle + \text{c.p.} = 0. \end{aligned}$$

Thus, when \mathcal{O} is invertible, we have a triple of compatible Hamiltonian structures on \mathcal{G}^* , of u -degrees zero, one, and two. When \mathcal{O} is not invertible, we are left with only linear and quadratic Poisson brackets.

To see that the quadratic Poisson bracket is *natural*, let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ be a homomorphism of Lie algebras. Let $\mathcal{O}_{\mathcal{H}} = \varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger : \mathcal{H}^* \rightarrow \mathcal{H}$ be the \mathcal{O} -operator induced on \mathcal{H}^* by the \mathcal{O} -operator $\mathcal{O} = \mathcal{O}_{\mathcal{G}}$ on \mathcal{G}^* . Let $\Phi : C_u \rightarrow C_q$ (5.20) be the corresponding homomorphism of function rings:

$$\Phi(\mathbf{u}) = \varphi^\dagger(\mathbf{q}). \quad (6.12)$$

To show that the map Φ is Hamiltonian between the quadratic Poisson brackets on \mathcal{G}^* and \mathcal{H}^* , we use Proposition 5.24. So, let H and F be two linear Hamiltonians in C_u :

$$H = \mathbf{Y}^t \mathbf{u}, \quad F = \mathbf{Z}^t \mathbf{u}, \quad \mathbf{Y}, \mathbf{Z} \in \mathcal{G}. \quad (6.13)$$

Then, by formulae (6.3) and (6.12),

$$\begin{aligned}
\Phi(\{H, F\}_{\mathcal{G}^*}) &\sim \Phi(\langle \mathbf{Y} \cdot \mathbf{u}, \mathcal{O}_{\mathcal{G}}(\mathbf{Z} \cdot \mathbf{u}) \rangle) = \langle \mathbf{Y} \cdot (\varphi^\dagger(\mathbf{q}), \mathcal{O}_{\mathcal{G}}(\mathbf{Z} \cdot \varphi^\dagger(\mathbf{q}))) \rangle \\
&\stackrel{[\text{by (4.15)}]}{=} \langle \varphi^\dagger(\varphi(\mathbf{Y}) \cdot \mathbf{q}), \mathcal{O}_{\mathcal{G}} \varphi^\dagger(\varphi(\mathbf{Z}) \cdot \mathbf{q}) \rangle \sim \langle \varphi(\mathbf{Y}) \cdot \mathbf{q}, \varphi \mathcal{O}_{\mathcal{G}} \varphi^\dagger(\varphi(\mathbf{Z}) \cdot \mathbf{q}) \rangle \\
&= \langle \varphi(\mathbf{Y}) \cdot \mathbf{q}, \mathcal{O}_{\mathcal{H}}(\varphi(\mathbf{Z}) \cdot \mathbf{q}) \rangle = \left\langle \frac{\delta \Phi(H)}{\delta \mathbf{q}} \cdot \mathbf{q}, \mathcal{O}_{\mathcal{H}} \left(\frac{\delta \Phi(F)}{\delta \mathbf{q}} \cdot \mathbf{q} \right) \right\rangle \sim \{\Phi(H), \Phi(F)\}_{\mathcal{H}^*},
\end{aligned} \tag{6.14}$$

where we used in the next to last equality in this chain the formula

$$\frac{\delta \Phi(H)}{\delta \mathbf{q}} = \varphi(\mathbf{Y}),$$

which follows from the relations

$$\Phi(H) = \Phi(\langle \mathbf{u}, \mathbf{Y} \rangle) = \langle \varphi^\dagger(\mathbf{q}), \mathbf{Y} \rangle \sim \langle \mathbf{q}, \varphi(\mathbf{Y}) \rangle.$$

What are the Casimirs of the quadratic bracket? Formula (6.4) shows that they are precisely the solutions of the equation

$$\mathcal{O} \left(\frac{\delta H}{\delta \mathbf{u}} \cdot \mathbf{u} \right) \cdot \mathbf{u} = 0. \tag{6.15}$$

In particular, all “coadjoint invariants”, i.e., those H satisfying

$$\frac{\delta H}{\delta \mathbf{u}} \cdot \mathbf{u} = 0, \tag{6.16}$$

are Casimirs, so that, in finite dimensions, the symplectic leaves of the quadratic bracket sit inside the coadjoint orbits. A better understanding of the symplectic leaves should be interesting.

Let us now check the infinitesimal Hamiltonian action criterion for the quadratic bracket. By Theorem 5.39, we have to verify the relation (5.34) for linear Hamiltonians H and F given by formula (6.13). Starting with the RHS of the criterion (5.34) and using formulae (5.41) and (2.11), we obtain:

$$\begin{aligned}
\langle [H^\sim, F^\sim], X \rangle &= \langle [-Y \cdot u, -Z \cdot u], X \rangle \\
&= \langle \mathcal{O}(Y \cdot u) \cdot (Z \cdot u) - \mathcal{O}(Z \cdot u) \cdot (Y \cdot u), X \rangle = \langle \mathcal{O}(\bar{Y}) \cdot \bar{Z} - \mathcal{O}(\bar{Z}) \cdot \bar{Y}, X \rangle,
\end{aligned} \tag{6.17}$$

where we introduced the convenient notation

$$\bar{Y} = Y \cdot u, \quad \bar{Z} = Z \cdot u. \tag{6.18}$$

For LHS of the criterion (5.34) we use the form (5.44):

$$Z^t(X^\wedge(B)(Y) + Z^t B([X, Y]) + [X, Z]^t B(Y) \tag{6.19}$$

$$\begin{aligned}
&\stackrel{[\text{by (6.4)}]}{\sim} Z^t(\mathcal{O}(Y \cdot (X \cdot u)) \cdot u + \mathcal{O}(Y \cdot u) \cdot (X \cdot u)) - [X, Y]^t \mathcal{O}(Z \cdot u) \cdot u + [X, Z]^t \mathcal{O}(Y \cdot u) \cdot u \\
&\stackrel{[\text{by (6.18), (6.22)}]}{\sim} \langle -Z \cdot u, \mathcal{O}(Y \cdot (X \cdot u)) \rangle + \langle X \cdot u, [Z, \mathcal{O}(\bar{Y})] \rangle
\end{aligned} \tag{6.20}$$

$$+ \langle u, [\mathcal{O}(\bar{Z}), [X, Y]] \rangle - \langle u, [\mathcal{O}(\bar{Y}), [X, Z]] \rangle. \tag{6.21}$$

(We used above the obvious relation

$$X^t(Y \cdot u) \sim -Y^t(X \cdot u). \quad (6.22)$$

The 1st summand in the expression (6.20) can be transformed as

$$\langle Y \cdot (X \cdot u), \mathcal{O}(\bar{Z}) \rangle \sim \langle u, [X, [Y, \mathcal{O}(\bar{Z})]] \rangle, \quad (6.23a)$$

while the second summand in (6.20) is \sim to

$$- \langle u, [X, [Z, \mathcal{O}(\bar{Y})]] \rangle. \quad (6.23b)$$

Altogether, expression (6.21) and (6.23) add up to

$$\langle u, -[Y, [\mathcal{O}(\bar{Z}), X]] + [Z, [\mathcal{O}(\bar{Y}), X]] \rangle \sim \langle \mathcal{O}(\bar{Z}) \cdot (Y \cdot u) - \mathcal{O}(\bar{Y}) \cdot (Z \cdot u), X \rangle, \quad (6.24)$$

and this is the same as the expression (6.17).

Example 6.25. Let $\mathcal{G} = \mathcal{G}(\mu)$ be the Lie algebra (3.7),

$$\left[\begin{pmatrix} X \\ f \end{pmatrix}, \begin{pmatrix} Y \\ g \end{pmatrix} \right] = \begin{pmatrix} XY' - X'Y \\ (Xg - Yf + \mu(X'Y'' - X''Y'))' \end{pmatrix}, \quad (6.26)$$

and let \mathcal{O} be the \mathcal{O} -operator (3.12):

$$\mathcal{O} = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \partial^3 \end{pmatrix}. \quad (6.27)$$

By formula (3.14),

$$\begin{pmatrix} X \\ f \end{pmatrix} \cdot \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} Xu' + 2X'u - fp' + \mu(X'p'')'' + \mu(X''p')' \\ Xp' \end{pmatrix}. \quad (6.28)$$

For $H \in C_{u,p}$, let

$$\begin{pmatrix} X \\ f \end{pmatrix} = \begin{pmatrix} \delta H / \delta u \\ \delta H / \delta p \end{pmatrix}. \quad (6.29)$$

Then the motion equations for the Hamiltonian vector field X_H , by formula (6.4), are

$$\begin{aligned} \begin{pmatrix} u \\ p \end{pmatrix}_t &= X_H \begin{pmatrix} u \\ p \end{pmatrix} = B(\mathbf{X}) = B \begin{pmatrix} \delta H / \delta u \\ \delta H / \delta p \end{pmatrix} = \mathcal{O} \left(\mathbf{X} \cdot \begin{pmatrix} u \\ p \end{pmatrix} \right) \cdot \begin{pmatrix} u \\ p \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 & 1 \\ -1 & \epsilon \partial^3 \end{pmatrix} \begin{pmatrix} Xu' + 2X'u - fp' + \mu(X'p'')'' + \mu(X''p')' \\ Xp' \end{pmatrix} \right) \cdot \begin{pmatrix} u \\ p \end{pmatrix} \\ &= \begin{pmatrix} Xp' \\ \epsilon(Xp')''' - Xu' - 2X'u - \mu(X'p')'' - \mu(X''p')' + fp' \end{pmatrix} \cdot \begin{pmatrix} u \\ p \end{pmatrix} \\ &= \begin{pmatrix} Xp'u' + 2(Xp')'u + p'(-\epsilon(Xp')''' - fp' + Xu' + 2X'u + \mu(X'p'')'' \\ + \mu(X''p')' + \mu((Xp')'p')'' + \mu((Xp'')'p')' \\ \hline Xp'^2 \end{pmatrix}. \end{aligned} \quad (6.30)$$

Hence, the quadratic Hamiltonian matrix on $\mathcal{G}(\mu^*$ is

$$B = \begin{pmatrix} * & -p'^2 \\ p'^2 & 0 \end{pmatrix}, \quad (6.31)$$

$$* = 2(p'u\partial + \partial p'u) + (4\mu - \epsilon)p'\partial^3 p' + \mu(3p''^2 - 2p'p''')\partial + \mu\partial(3p''^2 - 2p'p'''). \quad (6.32)$$

This quadratic Hamiltonian matrix is compatible with the linear Hamiltonian matrix

$$B^{\text{lin}} = - \begin{pmatrix} ** & -p' \\ p' & 0 \end{pmatrix}, \quad (6.33a)$$

$$** = u\partial + \partial u + \mu(\partial^2 p'\partial + \partial p'\partial^2), \quad (6.33b)$$

and both these Hamiltonian matrices are compatible with the constant-coefficient Hamiltonian matrix

$$\mathcal{O}^{-1} = \begin{pmatrix} \epsilon\partial^3 & -1 \\ 1 & 0 \end{pmatrix}. \quad (6.34)$$

For $\mu = \epsilon = 0$, formulate (6.31)–(6.34) have n -dimensional analogs. See Appendix A2.

We conclude this Section by calculating the quadratic Poisson bracket in finite dimensions. Let (e_i) be a basis in \mathcal{G} , (e^i) the dual basis in \mathcal{G}^* , $r = \sum_{ij} r^{ij} e_i \otimes e_j \in \mathcal{G}^{\otimes 2}$, $r^{ij} = -r^{ji}$, the classical r -matrix, (c_{ij}^k) the structure constants of \mathcal{G} in the chosen basis. Then

$$\mathcal{O}(e^i) = \sum_s e_s r^{si}, \quad (6.35)$$

$$e_j e^j = - \sum_s c_{is}^j e^s, \quad (6.36)$$

$$\begin{aligned} \{u_i, u_j\} &= \langle e_i \mathbf{u}, \mathcal{O}(e_j \mathbf{u}) \rangle = \sum_{\alpha\beta} u_\alpha u_\beta \langle e_i e^\alpha, \mathcal{O}(e_j e_\beta) \rangle \\ &= \sum_{\alpha\beta k\ell} u_\alpha u_\beta c_{ik}^\alpha c_{j\ell}^\beta \langle e^k, \mathcal{O}(e^\ell) \rangle = \sum_{\alpha\beta k\ell} u_\alpha u_\beta c_{ik}^\alpha c_{j\ell}^\beta r^{k\ell}. \end{aligned} \quad (6.37)$$

This is formula (28) in [11]. All other results of this Section had been established for the finite-dimensional case in that paper.

7 Symplectic models for linear Poisson brackets on dual spaces to Lie algebras

Let $\chi : \mathcal{G} \rightarrow \text{Diff}(V)$ be a representation of a Lie algebra \mathcal{G} on a vector space V . Let

$$\nabla : V \times V^* \rightarrow \mathcal{G}^* \quad (7.1)$$

be the map defined by the relation

$$\langle v \nabla v^*, X \rangle \sim \langle v^*, \chi(X)(v) \rangle, \quad \forall v \in V, v^* \in V^*, X \in \mathcal{G}. \quad (7.2)$$

This map is then *Hamiltonian*, between the linear Poisson bracket on \mathcal{G}^* and symplectic Poisson bracket on $V \oplus V^*$. This was proven in [5] Ch. 8, where I called such maps Clebsch representations. We shall see in the next Section that the *same map* (7.1) is Hamiltonian between the quadratic Poisson bracket on \mathcal{G}^* defined in § 6 and some interesting *quadratic* Poisson bracket on $V \oplus V^*$. In this Section we prepare the ground for the next one, by fixing notation and quickly reproving the Hamiltonian property of the map ∇ (7.1) for the linear Poisson bracket.

Let $C_{\mathcal{M}} = \text{Fun}(V \oplus V^*) = R[x_{\alpha}^{(\sigma)}, p_{\alpha}^{(\sigma)}]$, $\alpha = 1, \dots, \dim(V)$. Define the symplectic Poisson bracket on $C_{\mathcal{M}}$ by the matrix

$$b = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (7.3)$$

so that

$$\{H, F\} \sim \begin{pmatrix} \frac{\delta F}{\delta \mathbf{x}} \\ \frac{\delta F}{\delta \mathbf{p}} \end{pmatrix}^t b \begin{pmatrix} \frac{\delta H}{\delta \mathbf{x}} \\ \frac{\delta H}{\delta \mathbf{p}} \end{pmatrix} = -\frac{\delta F}{\delta \mathbf{x}^t} \frac{\delta H}{\delta \mathbf{p}} + \frac{\delta F}{\delta \mathbf{p}^t} \frac{\delta H}{\delta \mathbf{x}}. \quad (7.4)$$

Let $\Phi : C_{\mathcal{G}}^* \rightarrow C_{\mathcal{M}}$ be the differential (-difference) homomorphism defined on the generators of the ring $C_{\mathcal{G}}^* = C_u = R[u_i^{(\sigma)}]$, $i = 1, \dots, \dim(\mathcal{G})$, by the rule

$$\Phi(\mathbf{u}) = \mathbf{x} \nabla \mathbf{p}. \quad (7.5)$$

To show that this map Φ is Hamiltonian, we appeal to Propositions 5.24, choose two linear in u Hamiltonians

$$H = \mathbf{u}^t \mathbf{Y}, \quad F = \mathbf{u}^t \mathbf{Z}, \quad \mathbf{Y}, \mathbf{Z} \in \mathcal{G}, \quad (7.6)$$

and then have:

$$\Phi(\{H, F\}_{\mathcal{G}^*}) \sim \Phi(\mathbf{u}^t [\mathbf{Y}, \mathbf{Z}]) = \langle \mathbf{x} \nabla \mathbf{p}, [\mathbf{Y}, \mathbf{Z}] \rangle \sim \langle \mathbf{p}, [\mathbf{Y}, \mathbf{Z}].\mathbf{x} \rangle, \quad (7.7)$$

$$\Phi(H) = \Phi(\mathbf{u}^t \mathbf{Y}) = \langle \mathbf{x} \nabla \mathbf{p}, \mathbf{Y} \rangle \sim \langle \mathbf{p}, \mathbf{Y}.\mathbf{x} \rangle \sim \langle -\mathbf{Y}.\mathbf{p}, \mathbf{x} \rangle \quad (7.8)$$

$$\Rightarrow \frac{\delta \Phi(H)}{\delta \mathbf{x}} = -\mathbf{Y}.\mathbf{p}, \quad \frac{\delta \Phi(H)}{\delta \mathbf{p}} = \mathbf{Y}.\mathbf{x} \quad (7.9)$$

$$\begin{aligned} \Rightarrow \{ \Phi(H), \Phi(F) \}_{\mathcal{M}} &\stackrel{[\text{by } (7.4)]}{\sim} (\mathbf{Z}.\mathbf{p})^t (\mathbf{Y}.\mathbf{x}) - (\mathbf{Z}.\mathbf{x})^t (\mathbf{Y}.\mathbf{p}) \\ &\sim \langle \mathbf{p}, -\mathbf{Z}.\mathbf{p} \rangle + \langle \mathbf{p}, \mathbf{Y}.\mathbf{x} \rangle = \langle \mathbf{p}, -\mathbf{Z}.\mathbf{p} + \mathbf{Y}.\mathbf{x} \rangle \\ &= \langle \mathbf{p}, [\mathbf{Y}, \mathbf{Z}].\mathbf{x} \rangle, \end{aligned} \quad (7.10)$$

and this is the same as the expression (7.7).

Remark 7.11. Strictly speaking, we are dealing here not with the *full* Clebsch representations, – which are Hamiltonian maps: $\text{Fun}((\mathcal{G} \times V^*) \rightarrow \text{Fun}(V \oplus V^*))$, – but only with their \mathcal{G}^* -components, in which the nontriviality of results resides; and in any case, this component is the only one we need in this paper.

8 Clebsch representations for quadratic Poisson brackets on dual spaces to Lie algebras

For the linear Hamiltonian $H, F \in C_{\mathcal{G}^*} = C_u$,

$$H = \mathbf{u}^t \mathbf{Y}, \quad F = \mathbf{u}^t \mathbf{Z}, \quad (8.1)$$

The quadratic Poisson brackets (6.3) on \mathcal{G}^* yields:

$$\{H, F\}_{\mathcal{G}^*} \sim \langle \mathbf{Y} \cdot \mathbf{u}, \mathcal{O}(\mathbf{Z} \cdot \mathbf{u}) \rangle. \quad (8.2)$$

Therefore, the image under the map $\Phi : \mathcal{O}_{\mathcal{G}^*} \rightarrow C_{\mathcal{M}}$ of the Poisson bracket $\{H, F\}_{\mathcal{G}^*}$ is:

$$\Phi(\{H, F\}_{\mathcal{G}^*}) \sim \langle \mathbf{Y} \cdot (\mathbf{x} \nabla \mathbf{p}), \mathcal{O}(\mathbf{Z} \cdot (\mathbf{x} \nabla \mathbf{p})) \rangle. \quad (8.3)$$

On the other hand, by formula (7.9),

$$\{\Phi(H), \Phi(F)\}_{\mathcal{M}} \sim \begin{pmatrix} -\mathbf{Z} \cdot \mathbf{p} \\ \mathbf{Z} \cdot \mathbf{x} \end{pmatrix}^t \begin{pmatrix} B_{xx} & B_{xp} \\ B_{px} & B_{pp} \end{pmatrix} \begin{pmatrix} -\mathbf{Y} \cdot \mathbf{p} \\ \mathbf{Y} \cdot \mathbf{x} \end{pmatrix}, \quad (8.4)$$

where

$$B = \begin{pmatrix} B_{xx} & B_{xp} \\ B_{px} & B_{pp} \end{pmatrix} \quad (8.5)$$

is the following *quadratic* Poisson bracket on $V \oplus V^*$:

$$(\{M^t \mathbf{x}, N^t \mathbf{x}\} \sim) N^t B_{xx}(M) \sim \langle \mathbf{x} \nabla M, \mathcal{O}(\mathbf{x} \nabla N) \rangle, \quad (8.6a)$$

$$(\{M^t \mathbf{p}, N^t \mathbf{x}\} \sim) N^t B_{xp}(M) \sim -\langle M \nabla \mathbf{p}, \mathcal{O}(\mathbf{x} \nabla N) \rangle, \quad (8.6b)$$

$$(\{M^t \mathbf{x}, N^t \mathbf{p}\} \sim) N^t B_{px}(M) \sim -\langle \mathbf{x} \nabla M, \mathcal{O}(N \nabla \mathbf{p}) \rangle, \quad (8.6c)$$

$$(\{M^t \mathbf{p}, N^t \mathbf{p}\} \sim) N^t B_{pp}(M) \sim \langle M \nabla \mathbf{p}, \mathcal{O}(N \nabla \mathbf{p}) \rangle, \quad (8.6d)$$

The formulae (8.6b) and (8.6c) obviously agree with each other; the matrix B (8.5) is thus skewsymmetric. In the next Section we shall verify that the corresponding quadratic Poisson bracket on $V \oplus V^*$ satisfies the Jacobi identity and is compatible with the symplectic Poisson bracket (7.4).

Let us check now that the map $\Phi : C_{\mathcal{G}^*} \rightarrow C_{\mathcal{M}}$, $\Phi(\mathbf{u}) = \mathbf{x} \nabla \mathbf{p}$, is Hamiltonian. Writing in long hand the expression (8.4) and using formulae (8.6), we get

$$\begin{aligned} \{\Phi(H), \Phi(F)\}_{\mathcal{M}} &\sim (-\mathbf{Z} \cdot \mathbf{p})^t B_{xx}(-\mathbf{Y} \cdot \mathbf{p}) + (-\mathbf{Z} \cdot \mathbf{p})^t B_{xp}(\mathbf{Y} \cdot \mathbf{x}) \\ &\quad + (\mathbf{Z} \cdot \mathbf{x})^t B_{px}(-\mathbf{Y} \cdot \mathbf{p}) + (\mathbf{Z} \cdot \mathbf{x})^t B_{pp}(\mathbf{Y} \cdot \mathbf{x}) \\ &\sim \langle \mathbf{x} \nabla(\mathbf{Y} \cdot \mathbf{p}), \mathcal{O}(\mathbf{x} \nabla(\mathbf{Z} \cdot \mathbf{p})) \rangle + \langle (\mathbf{Y} \cdot \mathbf{x}) \nabla \mathbf{p}, \mathcal{O}(\mathbf{x} \nabla(\mathbf{Z} \cdot \mathbf{p})) \rangle \\ &\quad + \langle \mathbf{x} \nabla(\mathbf{Y} \cdot \mathbf{p}), \mathcal{O}((\mathbf{Z} \cdot \mathbf{x}) \nabla \mathbf{p}) \rangle + \langle (\mathbf{Y} \cdot \mathbf{x}) \nabla \mathbf{p}, \mathcal{O}((\mathbf{Z} \cdot \mathbf{x}) \nabla \mathbf{p}) \rangle \\ &= \langle \mathbf{x} \nabla(\mathbf{Y} \cdot \mathbf{p}) + (\mathbf{Y} \cdot \mathbf{x}) \nabla \mathbf{p}, \mathcal{O}(\mathbf{x} \nabla(\mathbf{Z} \cdot \mathbf{p}) + (\mathbf{Z} \cdot \mathbf{x}) \nabla \mathbf{p}) \rangle \\ &\stackrel{[\text{by (8.8)}]}{=} \langle \mathbf{Y} \cdot (\mathbf{x} \nabla \mathbf{p}), \mathcal{O}(\mathbf{Z} \cdot (\mathbf{x} \nabla \mathbf{p})) \rangle \stackrel{[\text{by (8.3)}]}{=} \Phi(\{H, F\}_{\mathcal{G}^*}). \end{aligned}$$

Lemma 8.7.

$$X \cdot (v \nabla v^*) = (X.v) \nabla v^* + v \nabla (X.v^*), \quad \forall v \in V, v^* \in V^*, X \in \mathcal{G}. \quad (8.8)$$

Proof. For any $L \in \mathcal{G}$, we have

$$\begin{aligned} \langle X \cdot (v \nabla v^*), L \rangle &\sim \langle v \nabla v^*, [L, X] \rangle \sim \langle v^*, [L, X].v \rangle \\ &= \langle v^*, L.(X.v) - X.(L.v) \rangle \sim \langle (X.v) \nabla v^*, L \rangle + \langle v \nabla (X.v^*), L \rangle \\ &= \langle (X.v) \nabla v^* + v \nabla (X.v^*), L \rangle. \end{aligned} \quad \blacksquare$$

Remark 8.8. Originally, Clebsch representation was discovered by Clebsch in vector calculus on \mathbf{R}^n , $n = 2, 3$, without any Lie-algebraic connections. The later are the results of more recent developments, in 1980's, and are summarized and developed in my book [6]. Since the publications of that book in 1992, there have been 2.5 other developments I'm aware of. First, I found (in [8] § 6) *quantum* Clebsch representations for the linear Poisson brackets on the dual spaces to *finite-dimensional* Lie algebras. Second, for the general non-quantal case, the theory of Clebsch representations has been generalized into the noncommutative realm in [10]. Finally, Dr. Morrison in [12] pp. 500–503 independently published a simple version of some of the Clebsch representations results from [5], but in a slightly different notation.

We conclude this Section by writing down explicit quadratic Poisson bracket formulae on $V \oplus V^*$ for the case of finite dimensions. In the notation (6.35)–(6.37), let $\{\ell_\alpha\}$ be a basis in V , $\{\ell^\alpha\}$ the dual basis in V^* , and

$$\chi(e_i)(\ell_\alpha) = \sum_\gamma \chi_{i\alpha}^\gamma \ell_\gamma \quad (8.9)$$

be the action formulae for the representation $\chi : \mathcal{G} \rightarrow \text{End}(V)$. Then formulae (6.6) yield

$$\{x^\alpha, x^\beta\} = \sum_{st\mu\nu} r^{st} \chi_{s\mu}^\alpha \chi_{t\nu}^\beta x^\mu x^\nu, \quad (8.10a)$$

$$\{p_\alpha, x^\beta\} = - \sum_{st\mu\nu} r^{st} \chi_{s\alpha}^\mu \chi_{t\nu}^\beta p_\mu x^\nu, \quad (8.10b)$$

$$\{p_\alpha, p_\beta\} = \sum_{st\mu\nu} r^{st} \chi_{s\alpha}^\mu \chi_{t\beta}^\nu p_\mu p_\nu. \quad (8.10c)$$

The map $\Phi(\mathbf{u}) = \mathbf{x} \nabla \mathbf{p}$ takes the form

$$\Phi(u_s) = \sum_{\alpha\beta} \chi_{s\alpha}^\beta x^\alpha p_\beta, \quad (8.11)$$

and it is a Hamiltonian map between the quadratic Poisson bracket (6.37) on \mathcal{G}^* ,

$$\{u_i, u_j\} = \sum_{st\kappa\ell} r^{st} c_{is}^\kappa c_{jt}^\ell u_\kappa u_\ell, \quad (8.12)$$

and the quadratic Poisson brackets (8.10) on $V \oplus V^*$.

From the results in the next Section it follows that the quadratic Poisson brackets (8.10) on $V \oplus V^*$ satisfy the Jacobi identity and are compatible with the symplectic Poisson bracket

$$\{x^\alpha, p_\beta\} = \delta_\beta^\alpha, \quad \{x^\alpha, x^\beta\} = \{p_\alpha, p_\beta\} = 0. \quad (8.13)$$

Finite-dimensional formulae (8.10) can be found in Zakrzewski's paper [15].

9 Properties of the quadratic Poisson brackets on $V \oplus V^*$

In this Section we prove that: 1) the quadratic Poisson bracket (8.6) on $\text{Fun}(V \oplus V^*)$, induced by an \mathcal{O} -operator $\mathcal{O} : \mathcal{G}^* \rightarrow \mathcal{G}$ and a representation $\chi : \mathcal{G} \rightarrow \text{Diff}(V)$, is legitimate, i.e., it satisfies the Jacobi identity; 2) this quadratic Poisson bracket is compatible with the symplectic one; 3) the natural action of \mathcal{G} on $\text{Fun}(V \oplus V^*)$ satisfies the infinitesimal Hamiltonian action criterion (5.34) for this quadratic Poisson bracket.

Proposition 9.1. *The quadratic Poisson brackets (8.6) on $\text{Fun}(V \oplus V^*)$ satisfy the Jacobi identity.*

Proof. By the main Theorem of the Hamiltonian formalism, we have to verify that

$$\{\{H, F\}, G\} + \text{c.p.} \sim 0 \quad (9.2)$$

for all Hamiltonians H, F, G linear in the x 's and the p 's. We break the verification procedure into 4 cases indexed by the number of the p 's involved in H, F, G : zero, one, two, or three.

Case zero:

$$H = \mathbf{X}^t \mathbf{x}, \quad F = \mathbf{Y}^t \mathbf{x}, \quad G = \mathbf{Z}^t \mathbf{x}, \quad (9.3)$$

where, as understood throughout this paper, $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are arbitrary vectors with entries in R (or $\tilde{R} \supset R$). By formula (8.6a),

$$\{H, F\} \sim \langle \mathbf{x} \nabla \mathbf{X}, \mathcal{O}(\mathbf{x} \nabla \mathbf{Y}) \rangle. \quad (9.4)$$

Denoting temporarily

$$\bar{X} = \mathbf{x} \nabla \mathbf{X}, \quad \bar{\bar{X}} = \mathcal{O}(\bar{X}), \quad (9.5)$$

we get from formula (9.4) and the relations

$$\langle \mathbf{x} \nabla \mathbf{X}, \bar{Y} \rangle \sim \langle \mathbf{X}, \bar{Y} \cdot \mathbf{x} \rangle \sim -\langle \bar{Y} \cdot \mathbf{X}, \mathbf{x} \rangle \quad (9.6)$$

that

$$\frac{\delta}{\delta \mathbf{x}}(\{H, F\}) \sim -\bar{Y} \cdot \mathbf{X} + \bar{\bar{X}} \cdot \mathbf{Y}. \quad (9.7)$$

Therefore, by formula (8.6a) again,

$$\{\{H, F\}, G\} \sim \mathbf{Z}^t B_{xx}(\bar{\bar{X}} \cdot \mathbf{Y} - \bar{Y} \cdot \mathbf{X}) \sim \langle \mathbf{x} \nabla(\bar{\bar{X}} \cdot \mathbf{Y} - \bar{Y} \cdot \mathbf{X}), \bar{\bar{Z}} \rangle, \quad (9.8)$$

so that

$$\begin{aligned} \{\{H, F\}, G\} + \text{c.p.} &\sim (\langle \mathbf{x} \nabla(\bar{\bar{X}} \cdot \mathbf{Y}), \bar{\bar{Z}} \rangle + \text{c.p.}) - (\langle \mathbf{x} \nabla(\bar{Y} \cdot \mathbf{X}), \bar{\bar{Z}} \rangle + \text{c.p.}) \\ &= (\langle \mathbf{x} \nabla(\bar{\bar{X}} \cdot \mathbf{Y}), \bar{\bar{Z}} \rangle + \text{c.p.}) - (\langle \mathbf{x} \nabla(\bar{\bar{Z}} \cdot \mathbf{Y}), \bar{\bar{X}} \rangle + \text{c.p.}) \\ &= (\langle \mathbf{x} \nabla(\bar{\bar{X}} \cdot \mathbf{Y}), \bar{\bar{Z}} \rangle - \langle \mathbf{x} \nabla(\bar{\bar{Z}} \cdot \mathbf{Y}), \bar{\bar{X}} \rangle) + \text{c.p.} \\ &\sim \langle -\bar{\bar{Z}} \cdot (\bar{\bar{X}} \cdot \mathbf{Y}) + \bar{\bar{X}} \cdot (\bar{\bar{Z}} \cdot \mathbf{Y}), \mathbf{x} \rangle + \text{c.p.} = \langle [\bar{\bar{X}}, \bar{\bar{Z}}] \cdot \mathbf{Y}, \mathbf{x} \rangle + \text{c.p.} \\ &\sim \langle \mathbf{x} \nabla \mathbf{Y}, [\bar{\bar{Z}}, \bar{\bar{X}}] \rangle + \text{c.p.} = \langle \bar{Y}, [\mathcal{O}(\bar{\bar{Z}}), \mathcal{O}(\bar{\bar{X}})] \rangle + \text{c.p.} \sim 0 \end{aligned}$$

by formula (2.6);

Case one:

$$H = \mathbf{X}^t \mathbf{x}, \quad F = \mathbf{Y}^t \mathbf{x}, \quad G = \mathbf{Z}^t \mathbf{p}. \quad (9.9)$$

By formulae (9.7) and (8.6c),

$$\{\{H, F\}, G\} \sim \mathbf{Z}^t B_{px}(\bar{\bar{X}} \cdot \mathbf{Y} - \bar{\bar{Y}} \cdot \mathbf{X}) \sim -\langle \mathbf{x} \nabla (\bar{\bar{X}} \cdot \mathbf{Y} - \bar{\bar{Y}} \cdot \mathbf{X}), \mathcal{O}(\mathbf{Z} \nabla \mathbf{p}) \rangle. \quad (9.10)$$

On the other hand,

$$\{F, G\} \sim \mathbf{Z}^t B_{px}(\mathbf{Y}) \sim -\langle \mathbf{x} \nabla \mathbf{Y}, \mathcal{O}(\mathbf{Z} \nabla \mathbf{p}) \rangle \sim \langle \mathcal{O}(\mathbf{Z} \nabla \mathbf{p}) \cdot \mathbf{Y}, \mathbf{x} \rangle \sim \langle \mathbf{p}, \bar{\bar{Y}} \cdot \mathbf{Z} \rangle.$$

Thus, in the notation

$$\underline{\underline{Z}} = \mathbf{Z} \nabla \mathbf{p}, \quad \underline{\underline{Z}} = \mathcal{O}(\underline{\underline{Z}}),$$

we have

$$\frac{\delta}{\delta \mathbf{x}}(\{F, G\}) = \underline{\underline{Z}} \cdot \mathbf{Y}, \quad \frac{\delta}{\delta \mathbf{p}}(\{F, G\}) = \bar{\bar{Y}} \cdot \mathbf{Z}. \quad (9.11)$$

Therefore, by formulae (8.6a,b),

$$\begin{aligned} \{\{F, G\}, H\} &\sim \mathbf{X}^t [B_{xx}(\underline{\underline{Z}} \cdot \mathbf{Y}) + B_{xp}(\bar{\bar{Y}} \cdot \mathbf{Z})] \\ &\sim \langle \mathbf{x} \nabla (\underline{\underline{Z}} \cdot \mathbf{Y}), \bar{\bar{X}} \rangle - \langle (\bar{\bar{Y}} \cdot \mathbf{Z}) \nabla \mathbf{p}, \bar{\bar{X}} \rangle. \end{aligned} \quad (9.12)$$

Interchanging X and Y in formula (9.12), we obtain

$$\begin{aligned} \{\{G, H\}, F\} &\sim -\{\{H, G\}, F\} \\ &\sim -\langle \mathbf{x} \nabla (\underline{\underline{Z}} \cdot \mathbf{X}), \bar{\bar{Y}} \rangle + \langle (\bar{\bar{X}} \cdot \mathbf{Z}) \nabla \mathbf{p}, \bar{\bar{Y}} \rangle. \end{aligned} \quad (9.13)$$

The 2nd summands in the expressions (9.12) and (9.13) combine into

$$\begin{aligned} \langle \mathbf{p}, -\bar{\bar{X}} \cdot (\bar{\bar{Y}} \cdot \mathbf{Z}) + \bar{\bar{Y}} \cdot (\bar{\bar{X}} \cdot \mathbf{Z}) \rangle &= \langle \mathbf{p}, [\bar{\bar{Y}}, \bar{\bar{X}}] \cdot \mathbf{Z} \rangle \\ &\sim \langle \mathbf{Z} \nabla \mathbf{p}, [\mathcal{O}(\bar{\bar{Y}}), \mathcal{O}(\bar{\bar{X}})] \rangle \stackrel{[by (2.10)]}{=} \langle \underline{\underline{Z}}, \mathcal{O}(\bar{\bar{Y}} \cdot \bar{\bar{X}} - \bar{\bar{X}} \cdot \bar{\bar{Y}}) \rangle \\ &\sim -\langle \bar{\bar{Y}} \cdot \bar{\bar{X}} - \bar{\bar{X}} \cdot \bar{\bar{Y}}, \underline{\underline{Z}} \rangle, \end{aligned} \quad (9.14)$$

while the 1st summands in the expressions (9.12) and (9.13) combine into

$$\begin{aligned} \langle \underline{\underline{Z}} \cdot \mathbf{Y}, \underline{\underline{X}} \cdot \mathbf{x} \rangle - \langle \mathbf{Z} \cdot \mathbf{x}, \underline{\underline{Y}} \cdot \mathbf{x} \rangle &\sim -\langle \mathbf{Y}, \underline{\underline{Z}} \cdot (\underline{\underline{X}} \cdot \mathbf{x}) \rangle + \langle \mathbf{X}, \underline{\underline{Z}} \cdot (\underline{\underline{Y}} \cdot \mathbf{x}) \rangle \\ &\sim -\langle (\bar{\bar{X}} \cdot \mathbf{x}) \nabla \mathbf{Y} + (\bar{\bar{Y}} \cdot \mathbf{x}) \nabla \mathbf{X}, \bar{\bar{Z}} \rangle. \end{aligned} \quad (9.15)$$

We see that the sum total of the expressions (9.10), (9.14), and (9.15) is ~ 0 provided

$$-\mathbf{x} \nabla (\bar{\bar{X}} \cdot \mathbf{Y} - \bar{\bar{Y}} \cdot \mathbf{X}) - \bar{\bar{Y}} \cdot \bar{\bar{X}} + \bar{\bar{X}} \cdot \bar{\bar{Y}} - (\underline{\underline{X}} \cdot \mathbf{x}) \nabla \mathbf{Y} + (\underline{\underline{Y}} \cdot \mathbf{x}) \nabla \mathbf{X} = 0,$$

and this is so by formulae (8.8) and (9.5);

Cases two and three follow from cases one and zero, respectively, once we notice that the quadratic Poisson bracket formulae (8.6) allow the symmetry

$$\mathbf{x} \mapsto \mathbf{p}, \quad \mathbf{p} \mapsto \mathbf{x}, \quad \chi \mapsto \chi^d, \quad (9.16)$$

where $\chi^d : \mathcal{G} \mapsto \text{Diff}(V^*)$ is the dual representation,

$$\chi^d(X) = -\chi(X)^\dagger, \quad \forall X \in \mathcal{G}. \quad (9.17)$$

This symmetry becomes obvious if we use the relation

$$a \nabla b = -b \nabla_d a \quad (9.18)$$

where ∇_d product is taken w.r.t. the dual representation χ^d :

$$\langle a \nabla b, X \rangle \sim \langle b, X \cdot a \rangle \sim \langle -X \cdot b, a \rangle = \langle a, -X \cdot b \rangle \sim \langle -b \nabla_d a, X \rangle, \quad \forall X \in \mathcal{G}. \quad (9.19)$$

Formulae (8.6a) and (8.6d) are interchanged under the symmetry (9.16), as are formulae (8.6b) and (8.6c). ■

Proposition 9.20. *The quadratic Poisson bracket (8.6) on $V \oplus V^*$ and the symplectic bracket (7.4) are compatible.*

Proof. We have to verify that

$$(\{\{H, F\}_1, G\}_2 + \{\{H, F\}_2, G\}_1) + \text{c.p.} \sim 0 \quad (9.21)$$

for all H, F, G linear in \mathbf{x}, \mathbf{p} , where $\{, \}_1$ denotes the symplectic Poisson bracket and $\{, \}_2$ denotes the quadratic one. This is obviously true for H, F, G linear in \mathbf{x} ; and also for H, F, G linear in \mathbf{p} .

Case one:

$$H = \mathbf{X}^t \mathbf{x}, \quad F = \mathbf{Y}^t \mathbf{x}, \quad G = \mathbf{Z}^t \mathbf{p}. \quad (9.22)$$

We have

$$\{H, F\}_1 = 0, \quad \{F, G\}_1 \sim -\mathbf{Y}^t \mathbf{Z}, \quad \{G, H\}_1 \sim \mathbf{Z}^t \mathbf{X}.$$

Thus,

$$\{\{(\cdot), (\cdot)\}_1, (\cdot \cdot)\}_2 = 0.$$

Now, by formula (9.7),

$$\frac{\delta\{H, F\}}{\delta \mathbf{x}} = \bar{\bar{X}} \cdot \mathbf{Y} - \bar{\bar{Y}} \cdot \mathbf{X}, \quad (9.23)$$

so that, by formulae (7.4),

$$\{\{H, F\}_2, G\}_1 \sim -\mathbf{Z}^t (\bar{\bar{X}} \cdot \mathbf{Y} - \bar{\bar{Y}} \cdot \mathbf{X}) \sim \mathbf{Y}^t (\bar{\bar{X}} \cdot \mathbf{Z}) - \mathbf{X}^t (\bar{\bar{Y}} \cdot \mathbf{Z}). \quad (9.24)$$

Next, by formula (9.11),

$$\frac{\delta\{F, G\}_2}{\delta \mathbf{p}} = \bar{\bar{Y}} \cdot \mathbf{Z},$$

so that

$$\{\{F, G\}_2, H\}_1 \sim \mathbf{X}^t(\bar{Y} \cdot \mathbf{Z}). \quad (9.25a)$$

Interchanging X and Y in the formula (9.25a), we obtain

$$\{\{G, H\}_2, F\}_1 \sim -\{\{H, G\}_2, F\}_1 \sim -\mathbf{Y}^t(\bar{X} \cdot \mathbf{Z}). \quad (9.25b)$$

Adding up the expression (9.24)–(9.26) we get zero.

Case two follows from the already established case one (9.22) by the application of the symmetry (9.16) accompanied by the replacement of the symplectic matrix b (7.3) by $-b$, – which doesn't affect the validity of the case one arguments. ■

Remark 9.26. The Proof of Proposition 9.1 could be reduced to only the case zero upon noticing that formulae (8.6b–d) are particular instances of the basic formula (8.6a) applied to the representation $\chi^{\text{new}} = \chi \oplus \chi^d$ on $V^{\text{new}} = V \oplus V^*$.

Proposition 9.27. *The natural (anti) action of \mathcal{G} on $C_{\mathcal{M}} = \text{Fun}(V \oplus V^*)$,*

$$X^\wedge(\mathbf{x}) = X \cdot \mathbf{x}, \quad X^\wedge(\mathbf{p}) = X \cdot \mathbf{p}, \quad (9.28)$$

satisfies the criterion (5.34) of infinitesimal Hamiltonian action w.r.t. the quadratic Poisson bracket (8.6) on $V \oplus V^$.*

Proof. We shall check formula (5.34) for the linear Hamiltonians H, F . We break this check into 3 cases, depending upon how many p 's are present among H and F .

Case zero:

$$H = \mathbf{Y}^t \mathbf{x}, \quad F = \mathbf{Z}^t \mathbf{x}. \quad (9.29)$$

We have, by formula (8.6a):

$$\begin{aligned} X^\wedge(\{H, F\}) &\sim X^\wedge(\langle \mathbf{x} \nabla \mathbf{Y}, \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle) \\ &= \langle (X \cdot \mathbf{x}) \nabla \mathbf{Y}, \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle + \langle \mathbf{x} \nabla \mathbf{Y}, \mathcal{O}((X \cdot \mathbf{x}) \nabla \mathbf{Z}) \rangle, \end{aligned} \quad (9.30)$$

$$- \{X^\wedge(H), F\} \sim -\{\mathbf{Y}^t(X \cdot \mathbf{x}), \mathbf{Z}^t \mathbf{x}\} \sim \langle \mathbf{x} \nabla (X \cdot \mathbf{Y}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle, \quad (9.31)$$

$$- \{H, X^\wedge(F)\} = -\{\mathbf{Y}^t \mathbf{x}, \mathbf{Z}^t(X \cdot \mathbf{x})\} \sim \langle \mathbf{x} \nabla \mathbf{Y}, \mathcal{O}((X \cdot \mathbf{x}) \nabla \mathbf{Z}) \rangle. \quad (9.32)$$

Adding up the expressions (9.30)–(9.32) and using formula (8.8), we find

$$\begin{aligned} X^\wedge(\{H, F\}) - \{X^\wedge(H), F\} - \{H, X^\wedge(F)\} \\ \sim \langle X \cdot (\mathbf{x} \nabla \mathbf{Y}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle + \langle \mathbf{x} \nabla \mathbf{Y}, \mathcal{O}(X \cdot (\mathbf{x} \nabla \mathbf{Z})) \rangle. \end{aligned} \quad (9.33)$$

On the other hand, formulae (9.28) imply that

$$X^\wedge(\mathbf{Y}^t \mathbf{x}) = \mathbf{Y}^t(X \cdot \mathbf{x}) \sim \langle \mathbf{x} \nabla \mathbf{Y}, X \rangle, \quad (9.34a)$$

$$X^\wedge(\mathbf{Y}^t \mathbf{p}) = \mathbf{Y}^t(X \cdot \mathbf{p}) \sim \langle -\mathbf{Y} \nabla \mathbf{p}, X \rangle, \quad (9.34b)$$

so that

$$(\mathbf{Y}^t \mathbf{x})^\sim = \mathbf{x} \nabla \mathbf{Y}, \quad (\mathbf{Y}^t \mathbf{p})^\sim = -\mathbf{Y} \nabla \mathbf{p}. \quad (9.35)$$

Therefore,

$$\begin{aligned} \langle [H^\sim, F^\sim], X \rangle &= \langle [\mathbf{x} \nabla \mathbf{Y}, \mathbf{x} \nabla \mathbf{Z}], X \rangle \\ &\stackrel{[\text{by (2.11)}]}{=} \langle \mathcal{O}(\mathbf{x} \nabla \mathbf{Y}).(\mathbf{x} \nabla \mathbf{Z}) - \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}).(\mathbf{x} \nabla \mathbf{Y}), X \rangle \\ &\sim \langle -X^\cdot(\mathbf{x} \nabla \mathbf{Z}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Y}) \rangle + \langle X^\cdot(\mathbf{x} \nabla \mathbf{Y}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle, \end{aligned} \quad (9.36)$$

and this expression is \sim to (9.33) since \mathcal{O} is skewsymmetric.

Case one:

$$H = \mathbf{Y}^t \mathbf{p}, \quad F = \mathbf{Z}^t \mathbf{x}. \quad (9.37)$$

We have, by formula (8.6b):

$$\begin{aligned} X^\wedge(\{H, F\}) &\sim X^\wedge(-\langle \mathbf{Y} \nabla \mathbf{p}, \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle) \\ &= -\langle \mathbf{Y} \nabla (X^\cdot \mathbf{p}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle - \langle \mathbf{Y} \nabla \mathbf{p}, \mathcal{O}((X^\cdot \mathbf{x}) \nabla \mathbf{Z}) \rangle, \end{aligned} \quad (9.38)$$

$$\begin{aligned} -\{X^\wedge(H), F\} &= -\{\mathbf{Y}^t(X^\cdot \mathbf{p}), \mathbf{Z}^t \mathbf{x}\} \\ &\sim \{(X^\cdot \mathbf{Y})^t \mathbf{p}, \mathbf{Z}^t \mathbf{x}\} \sim -\langle (X^\cdot \mathbf{Y}) \nabla \mathbf{p}, \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle, \end{aligned} \quad (9.39)$$

$$\begin{aligned} -\{H, X^\wedge(F)\} &= -\{\mathbf{Y}^t \mathbf{p}, \mathbf{Z}^t(X^\cdot \mathbf{x})\} \\ &\sim \{\mathbf{Y}^t \mathbf{p}, (X^\cdot \mathbf{Z})^t \mathbf{x}\} \sim -\langle \mathbf{Y} \nabla \mathbf{p}, \mathcal{O}(\mathbf{x} \nabla (X^\cdot \mathbf{Z})) \rangle. \end{aligned} \quad (9.40)$$

Adding the expressions (9.38)–(9.40) up and using formula (8.8), we find

$$\begin{aligned} X^\wedge(\{H, F\}) - \{X^\wedge(H), F\} - \{H, X^\wedge(F)\} \\ \sim -\langle \mathbf{Y} \nabla \mathbf{p}, \mathcal{O}(X^\cdot(\mathbf{x} \nabla \mathbf{Z})) \rangle - \langle X^\cdot(\mathbf{Y} \nabla \mathbf{p}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle. \end{aligned} \quad (9.41)$$

On the other hand, by formulae (9.35) and (2.11),

$$\begin{aligned} \langle [H^\sim, F^\sim], X \rangle &= -\langle [\mathbf{Y} \nabla \mathbf{p}, \mathbf{x} \nabla \mathbf{Z}], X \rangle = -\langle \mathcal{O}(\mathbf{Y} \nabla \mathbf{p}).(\mathbf{x} \nabla \mathbf{Z}), X \rangle \\ &\quad + \langle \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}).(\mathbf{Y} \nabla \mathbf{p}), X \rangle \sim \langle X^\cdot(\mathbf{x} \nabla \mathbf{Z}), \mathcal{O}(\mathbf{Y} \nabla \mathbf{p}) \rangle - \langle X^\cdot(\mathbf{Y} \nabla \mathbf{p}), \mathcal{O}(\mathbf{x} \nabla \mathbf{Z}) \rangle, \end{aligned} \quad (9.42)$$

and this is \sim to the expression (9.41) because \mathcal{O} is skewsymmetric.

Case two follows from the case zero and the symmetry property (9.16). ■

Appendix A1. Crossed Lie algebras

Let G be a Hamilton-Lie group, i.e., a Lie group with a multiplicative Poisson bracket on it. The infinitesimal version of this object is a Lie bialgebra, i.e., a Lie bracket $[\cdot, \cdot]$ on \mathcal{G}^* whose dual, considered as a map $\varphi : \mathcal{G} \rightarrow \wedge^2 \mathcal{G}$, is 1-cocycle on \mathcal{G} . Drinfel'd noticed in [2] that the 1-cocycle condition can be reformated in such a way as to make the self-duality of the notion of Lie bialgebra explicit, as follows. Since \mathcal{G} is a Lie algebra, it acts on its dual space, \mathcal{G}^* . Consider the following skew multiplication on the space $\mathcal{G} + \mathcal{G}^*$:

$$\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right] = \begin{pmatrix} [X, Y] + u^\cdot Y - v^\cdot X \\ [u, v] + X^\cdot v - Y^\cdot u \end{pmatrix}, \quad X, Y \in \mathcal{G}, \quad u, v \in \mathcal{G}^*. \quad (\text{A1.1})$$

The Drinfel'd observation mentioned above is that the bracket (A1.1) satisfies the Jacobi identity iff $\varphi : \mathcal{G} \rightarrow \wedge^2 \mathcal{G}$ is a 1-cocycle on \mathcal{G} (see [1] p. 27). Let us find the form of the condition, originally written down by Drinfel'd in [2] for the finite-dimensional case, equivalent to formula (A1.1) defining a Lie algebra.

For the \mathcal{G} -component of the expression

$$\left[\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right], \begin{pmatrix} Z \\ w \end{pmatrix} \right] + \text{c.p.} = \left[\begin{pmatrix} [X, Y] + u \cdot Y - v \cdot X \\ [u, v] + X \cdot v - Y \cdot u \end{pmatrix}, \begin{pmatrix} Z \\ w \end{pmatrix} \right] + \text{c.p.} \quad (\text{A1.2})$$

we get:

$$([X, Y], Z] + \text{c.p.}) + ([u \cdot Y - v \cdot X, Z] + \text{c.p.}) + ([u, v] \cdot Z + \text{c.p.}) \quad (\text{A1.3a})$$

$$+ ([X \cdot v - Y \cdot u] \cdot Z - w \cdot [X, Y] + \text{c.p.}) - (w \cdot (u \cdot Y - v \cdot X) + \text{c.p.}). \quad (\text{A1.3b})$$

The first summand vanishes by the Jacobi identity in \mathcal{G} . The 3^{rd} and 5^{th} summands combine into

$$[w, u] \cdot Y - w \cdot (u \cdot Y) + u \cdot (w \cdot Y) + \text{c.p.}$$

and this vanishes because the action of \mathcal{G}^* on \mathcal{G} is a representation of the Lie algebra structure on \mathcal{G}^* . This leaves us with 2^{nd} and 4^{th} summands, which combine into

$$[u \cdot Y, Z] - [u \cdot Z, Y] + (Z \cdot u) \cdot Y - (Y \cdot u) \cdot Z - u \cdot [Y, Z] = 0. \quad (\text{A1.4})$$

This is an equation in \mathcal{G} . Let us calculate the value $\langle v, LHS \rangle$ for an arbitrary element $v \in \mathcal{G}^*$. Term-by-term, we find:

$$\langle v, [u \cdot Y, Z] \rangle \sim \langle Z \cdot v, u \cdot Y \rangle, \quad (\text{A1.5a})$$

$$\langle v, -[u \cdot Z, Y] \rangle \sim -\langle Y \cdot v, u \cdot Z \rangle, \quad (\text{A1.5b})$$

$$\langle v, (Z \cdot u) \cdot Y \rangle \sim \langle [v, Z \cdot u], Y \rangle \sim -\langle Z \cdot u, v \cdot Y \rangle, \quad (\text{A1.5c})$$

$$\langle v, -(Y \cdot u) \cdot Z \rangle \sim \langle [Y \cdot u, v], Z \rangle \sim -\langle Y \cdot u, v \cdot Z \rangle, \quad (\text{A1.5d})$$

$$\langle v, -u \cdot [Y, Z] \rangle \sim \langle [u, v], [Y, Z] \rangle. \quad (\text{A1.5e})$$

Adding the expressions (A1.5) up, we arrive at the following quadrilinear relation equivalent to the trilinear equation (A1.4):

$$\begin{aligned} \langle [u, v], [Y, Z] \rangle &\sim \langle Z \cdot v, u \cdot Y \rangle + \langle Y \cdot u, v \cdot Z \rangle - \langle Y \cdot v, u \cdot Z \rangle - \langle Z \cdot u, v \cdot Y \rangle, \\ \forall Y, Z \in \mathcal{G}, \quad \forall u, v \in \mathcal{G}^*. \end{aligned} \quad (\text{A1.6})$$

In this form, the symmetry between \mathcal{G} and \mathcal{G}^* is apparent. We don't have to analyze the \mathcal{G}^* -component of the double-commutator (A1.2).

Remark A1.7. In finite dimensions, the commutator (A1.1) leaves the natural scalar product on $\mathcal{G} + \mathcal{G}^*$:

$$\left(\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right) = \langle u, Y \rangle + \langle v, X \rangle, \quad (\text{A1.8})$$

ad-invariant:

$$\left(\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right], \begin{pmatrix} Z \\ w \end{pmatrix} \right) = \left(\begin{pmatrix} X \\ u \end{pmatrix}, \left[\begin{pmatrix} Y \\ v \end{pmatrix}, \begin{pmatrix} Z \\ w \end{pmatrix} \right] \right). \quad (\text{A1.9})$$

This ad-invariance is still true in infinite dimensions, provided it is properly understood:

$$\left(\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right], \begin{pmatrix} Z \\ w \end{pmatrix} \right) \sim \left(\begin{pmatrix} X \\ u \end{pmatrix}, \left[\begin{pmatrix} Y \\ v \end{pmatrix}, \begin{pmatrix} Z \\ w \end{pmatrix} \right] \right). \quad (\text{A1.10})$$

Indeed, for the LHS of (A1.10) we get

$$\begin{aligned} & \left(\begin{pmatrix} [X, Y] + u \cdot Y - v \cdot X \\ [u, v] + X \cdot v - Y \cdot u \end{pmatrix}, \begin{pmatrix} Z \\ w \end{pmatrix} \right) \\ &= \langle [u, v] + X \cdot u - Y \cdot u, Z \rangle + \langle w, [X, Y] + u \cdot Y - v \cdot X \rangle \\ &\sim \langle u, v \cdot Z \rangle - \langle Z \cdot v, X \rangle + \langle u, [Y, Z] \rangle + \langle Y \cdot w, X \rangle - \langle u, w \cdot Y \rangle + \langle [v, w], X \rangle \\ &= \langle u, v \cdot Z + [Y, Z] - w \cdot Y \rangle + \langle -Z \cdot v + Y \cdot w + [v, w], X \rangle \\ &= \left(\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} [Y, Z] + v \cdot Z - w \cdot Y \\ [v, w] + Y \cdot w - Z \cdot v \end{pmatrix} \right), \end{aligned}$$

and this is the RHS of (A1.10).

Let us check that when the commutator on \mathcal{G}^* is given by the formula (2.11),

$$[u, v] = \mathcal{O}(u) \cdot v - \mathcal{O}(v) \cdot u, \quad (\text{A1.11})$$

where $\mathcal{O} : \mathcal{G}^* \rightarrow \mathcal{G}$ is a skewsymmetric \mathcal{O} -operator, then $\mathcal{O} + \mathcal{O}^*$ (A1.1) is a Lie algebra. Let us check the criterion (A1.4).

Lemma A1.12.

$$u \cdot X = [\mathcal{O}(u), X] + \mathcal{O}(X \cdot u), \quad \forall u \in \mathcal{G}^*, X \in \mathcal{G}. \quad (\text{A1.13})$$

Proof. For any $v \in \mathcal{G}^*$,

$$\begin{aligned} \langle v, u \cdot X \rangle &\sim \langle [v, u], X \rangle = \langle \mathcal{O}(v) \cdot u - \mathcal{O}(u) \cdot v, X \rangle \sim \langle u, [X, \mathcal{O}(v)] \rangle + \langle v, [\mathcal{O}(u), X] \rangle \\ &\sim -\langle X \cdot u, \mathcal{O}(v) \rangle + \langle v, [\mathcal{O}(u), X] \rangle \sim \langle v, \mathcal{O}(X \cdot u) \rangle + \langle v, [\mathcal{O}(u), X] \rangle. \quad \blacksquare \end{aligned}$$

Using formula (A1.13), we transform each of the 5 summands in the LHS of the criterion (A1.4):

$$1) \quad [u \cdot Y, Z] = [[\mathcal{O}(u), Y], Z] + [\mathcal{O}(Y \cdot u), Z], \quad (\text{A1.14})$$

$$2) \quad -[u \cdot Z, Y] = -[[\mathcal{O}(u), Z], Y] - [\mathcal{O}(Z \cdot u), Y], \quad (\text{A1.15})$$

$$3) \quad (Z \cdot u) \cdot Y = [\mathcal{O}(Z \cdot u), Y] + \mathcal{O}(Y \cdot (Z \cdot u)), \quad (\text{A1.16})$$

$$4) \quad -(Y \cdot u) \cdot Z = -[\mathcal{O}(Y \cdot u), Z] - \mathcal{O}(Z \cdot (Y \cdot u)), \quad (\text{A1.17})$$

$$5) \quad -u \cdot [Y, Z] = -[\mathcal{O}(u), [Y, Z]] - \mathcal{O}([Y, Z] \cdot u). \quad (\text{A1.18})$$

The 2^{nd} summands in the expressions (A1.16)–(A1.18) add up to zero; the 1^{st} summands in the expressions (A1.14), (A1.15), (A1.18) do likewise; the 2^{nd} summand in the expression (A1.14+ ℓ) and the 1^{st} summand in the expression (A1.17+ ℓ), $\ell = 0, 1$, cancel each other out.

We conclude this Section by examining when the symplectic form on $\mathcal{G} + \mathcal{G}^*$:

$$\omega \left(\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right) = \langle u, Y \rangle - \langle v, X \rangle \quad (\text{A1.19})$$

is a 2-cocycle on the Lie algebra $\mathcal{G} + \mathcal{G}^*$ (A1.1). We have:

$$\begin{aligned} \omega \left(\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right], \begin{pmatrix} Z \\ w \end{pmatrix} \right) + \text{c.p.} &= \omega \left(\begin{pmatrix} [X, Y] + u \cdot Y - v \cdot X \\ [u, v] + X \cdot v - Y \cdot u \end{pmatrix}, \begin{pmatrix} Z \\ w \end{pmatrix} \right) + \text{c.p.} \\ &= (\langle [u, v] + X \cdot v - Y \cdot u, Z \rangle + \text{c.p.}) - (\langle w, [X, Y] + u \cdot Y - v \cdot X \rangle + \text{c.p.}) \\ &= (\langle [u, v], Z \rangle - \langle u, v \cdot Z \rangle + \langle v, u \cdot Z \rangle) + \text{c.p.} \end{aligned} \quad (\text{A1.20})$$

$$+ (\langle Y \cdot w, X \rangle - \langle X \cdot w, Y \rangle - \langle w, [X, Y] \rangle) + \text{c.p.} \quad (\text{A1.21})$$

Thus, the symplectic form (A1.19) is a 2-cocycle iff

$$\langle [u, v], Z \rangle - \langle u, v \cdot Z \rangle + \langle v, u \cdot Z \rangle \sim 0, \quad \forall u, v \in \mathcal{G}^*, Z \in \mathcal{G}, \quad (\text{A1.22})$$

$$\langle Y \cdot w, X \rangle - \langle X \cdot w, Y \rangle - \langle w, [X, Y] \rangle \sim 0, \quad \forall X, Y \in \mathcal{G}, w \in \mathcal{G}^*. \quad (\text{A1.23})$$

This happens iff, respectively,

$$[u, v] = 0, \quad \forall u, v \in \mathcal{G}^*, \quad (\text{A1.24})$$

$$[X, Y] = 0, \quad \forall X, Y \in \mathcal{G}, \quad (\text{A1.25})$$

which is this side of “never”. The next Section provides a remedy of sorts.

Appendix A2. Symplectic r -matrices and symplectic doubles

Let \mathcal{G} be a Lie algebra and $\rho : \mathcal{G} \rightarrow \text{Diff}(\mathcal{G}^*)$ a representation, *not necessarily* the coadjoint one. Let $\mathcal{H} := \mathcal{G} \ltimes_{\rho} \mathcal{G}^* = \mathcal{G} \ltimes_{\rho} \mathcal{G}^*$ be the semidirect sum Lie algebra, with the commutator

$$\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right] = \begin{pmatrix} [X, Y] \\ \rho(X)(v) - \rho(Y)(u) \end{pmatrix}. \quad X, Y \in \mathcal{G}, \quad u, v \in \mathcal{G}^*. \quad (\text{A2.1})$$

Let ω be the symplectic form on $\mathcal{H} = \mathcal{G} \ltimes_{\rho} \mathcal{G}^*$:

$$\omega \left(\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right) = \langle u, Y \rangle - \langle v, X \rangle. \quad (\text{A2.2})$$

Proposition A2.3. *The symplectic form ω (A2.2) is a 2-cocycle on the Lie algebra $\mathcal{G} \ltimes_{\rho} \mathcal{G}^*$ iff the dual representation $\rho^d : \mathcal{G} \rightarrow \text{Diff}(\mathcal{G})$ satisfies the property*

$$\rho^d(X)(Y) - \rho^d(Y)(X) = [X, Y], \quad \forall X, Y \in \mathcal{G}. \quad (\text{A2.4})$$

Proof. We have,

$$\begin{aligned} \omega \left(\left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right], \begin{pmatrix} Z \\ w \end{pmatrix} \right) + \text{c.p.} &= \omega \left(\begin{pmatrix} [X, Y] \\ \rho(X)(v) - \rho(Y)(u) \end{pmatrix}, \begin{pmatrix} Z \\ w \end{pmatrix} \right) + \text{c.p.} \\ &= (\langle \rho(X)(v) - \rho(Y)(u), Z \rangle - \langle w, [X, Y] \rangle) + \text{c.p.} \\ &= (\langle \rho(Y)(w), X \rangle - \langle \rho(X)(w), Y \rangle - \langle w, [X, Y] \rangle) + \text{c.p.} \end{aligned} \quad (\text{A2.5})$$

Thus, ω is a 2-cocycle iff

$$\langle \rho(Y)(w), X \rangle - \langle \rho(X)(w), Y \rangle \sim \langle w, [X, Y] \rangle, \quad \forall X, Y \in \mathcal{G}, \quad w \in \mathcal{G}^*. \quad (\text{A2.6})$$

Rewriting the LHS of this relation as

$$\langle w, -\rho^d(Y)(X) \rangle + \langle w, \rho^d(X)(Y) \rangle,$$

we arrive at the equivalent to (A2.6) equation (A2.4). ■

Assuming from now on that a representation $\rho^d : \mathcal{G} \rightarrow \text{Diff}(\mathcal{G})$ satisfying (A2.4) is fixed, denote by

$$xy = \rho^d(x)(y) \quad (\text{A2.7})$$

the resulting multiplication $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$. Since ρ^d is a representation,

$$\rho^d([x, y]) = [\rho^d(x), \rho^d(y)], \quad \forall x, y \in \mathcal{G}. \quad (\text{A2.8})$$

Applying this operator identity to an element $z \in \mathcal{G}$, we get

$$(xy - yx)z = x(yz) - y(xz), \quad (\text{A2.9})$$

or

$$(xy)z - x(yz) = (yx)z - y(xz), \quad \forall x, y, z \in \mathcal{G}. \quad (\text{A2.10})$$

Thus, \mathcal{G} is a *quasiassociative algebra*, and $T^*\mathcal{G} := \mathcal{G} \times_{\rho} \mathcal{G}^*$ (A2.1) is a proper phase space of \mathcal{G} , meaning that the symplectic form ω is a 2-cocycle on $T^*\mathcal{G}$. This explains the appearance of quasiassociative algebras in [7, 9]. There are many examples of quasiassociative algebras given in [7, 9] (in addition to obvious ones coming from associative algebras) such as Lie algebras of vector fields on \mathbf{R}^n [7] and on $GL(n)$ [9]. We shall now use the former example $\mathcal{D}_n = \mathcal{D}(\mathbf{R}^n) = \{X \in R^n, R = C^\infty(\mathbf{R}^n)\}$, with the quasiassociative multiplication

$$(\mathbf{X}\mathbf{Y})^i = \sum_s X^s Y_{,s}^i \quad (\text{A2.11})$$

where

$$(\cdot)_{,s} = \partial_s(\cdot) = \frac{\partial(\cdot)}{\partial x^s}. \quad (\text{A2.12})$$

The quasiassociative property (A2.10) is satisfied because

$$(\mathbf{X}(\mathbf{Y}\mathbf{Z}) - (\mathbf{X}\mathbf{Y})\mathbf{Z})^i = \sum_{st} X^s Y^t Z_{,st}^i \quad (\text{A2.13})$$

is symmetric in X, Y .

Now, formula (A2.11) means that

$$\rho^d(\mathbf{X}) = \hat{X}\mathbf{1}, \quad \hat{X} = \sum_s X^s \partial_s. \quad (\text{A2.14})$$

Therefore, $\rho(X) = -\rho^d(X)^\dagger$ is:

$$\rho(X) = \left(\sum_s \partial_s X^s \right) \mathbf{1}. \quad (\text{A2.15})$$

Hence, the Lie bracket on the Lie algebra $\mathcal{D}_n^{(1)} = \mathcal{D}_n \ltimes \mathcal{D}_n^*$ is

$$\left[\begin{pmatrix} \mathbf{X} \\ \mathbf{u} \end{pmatrix}, \begin{pmatrix} \mathbf{Y} \\ \mathbf{v} \end{pmatrix} \right]_i^i = \sum_s \begin{pmatrix} X^s Y_{,s}^i - Y^s X_{,s}^i \\ (X^s v_i - Y^s u_i)_{,s} \end{pmatrix}. \quad (\text{A2.16})$$

By general theory, since the symplectic form ω (A2.2) is a *nondegenerate* 2-cocycle on $\mathcal{G}^{(1)} = \mathcal{G} \ltimes \mathcal{G}^*$, it can be represented by an \mathcal{O} -operator $\mathcal{O} : \mathcal{G}^{(1)*} \rightarrow \mathcal{G}^{(1)}$. Writing elements of $\mathcal{G}^{(1)}$ as $\begin{pmatrix} \alpha \\ a \end{pmatrix}$, $\alpha \in \mathcal{G}^*$, $a \in \mathcal{G}$, we see from formulae (A2.2) and (2.3) that

$$-\omega \left(\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right) = \langle \mathcal{O}^{-1} \begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \rangle = -\langle u, Y \rangle + \langle v, X \rangle, \quad (\text{A2.17})$$

whence

$$\mathcal{O}^{-1} \begin{pmatrix} X \\ u \end{pmatrix} = \begin{pmatrix} -u \\ X \end{pmatrix}, \quad (\text{A2.18})$$

so that

$$\mathcal{O}^{-1} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \quad (\text{A2.19})$$

and hence

$$\mathcal{O} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \quad (\text{A2.20})$$

The resulting Lie algebra bracket (2.11) on $\mathcal{G}^{(1)*}$ takes the form

$$\begin{aligned} \left[\begin{pmatrix} \alpha \\ a \end{pmatrix}, \begin{pmatrix} \beta \\ b \end{pmatrix} \right] &= \begin{pmatrix} a \\ -\alpha \end{pmatrix} \cdot \begin{pmatrix} \beta \\ b \end{pmatrix} - \begin{pmatrix} b \\ -\beta \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ a \end{pmatrix} \\ &\stackrel{[\text{by (A2.23)}]}{=} \begin{pmatrix} a\beta - \beta a - \alpha b \\ ab \end{pmatrix} - \begin{pmatrix} b\alpha - \alpha b - \beta a \\ ba \end{pmatrix} = \begin{pmatrix} a\beta - b\alpha \\ [a, b] \end{pmatrix}. \end{aligned} \quad (\text{A2.21})$$

Lemma A2.22. *In $\mathcal{G}^{(1)*}$, the coadjoint action formulae is*

$$\begin{pmatrix} X \\ u \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ a \end{pmatrix} = \begin{pmatrix} X\alpha - \alpha X + ua \\ Xa \end{pmatrix}, \quad X, a \in \mathcal{G}, \quad u, \alpha \in \mathcal{G}^*. \quad (\text{A2.23})$$

(The new notation is explained in formulae (A2.25) below.)

Proof. We have,

$$\begin{aligned}
 \left\langle \begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} \alpha \\ a \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right\rangle &\sim -\left\langle \begin{pmatrix} \alpha \\ a \end{pmatrix}, \left[\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} Y \\ v \end{pmatrix} \right] \right\rangle = -\left\langle \begin{pmatrix} \alpha \\ a \end{pmatrix}, \begin{pmatrix} [X, Y] \\ \rho(X)(v) - \rho(Y)(u) \end{pmatrix} \right\rangle \\
 &= -\langle \alpha, [X, Y] \rangle - \langle \rho(X)(v) - \rho(Y)(u), a \rangle \\
 &\sim \langle X \cdot \alpha, Y \rangle - \langle v, \rho(X)^\dagger(a) \rangle + \langle u, \rho(Y)^\dagger(a) \rangle = \langle X \cdot \alpha, Y \rangle + \langle v, Xa \rangle - \langle u, Ya \rangle.
 \end{aligned} \tag{A2.24}$$

Let us define the left and right multiplication of \mathcal{G} on \mathcal{G}^* by the relations

$$\langle Xu, Y \rangle \sim -\langle u, XY \rangle, \quad \forall u \in \mathcal{G}^*, X, Y \in \mathcal{G}, \tag{A2.25a}$$

$$\langle uX, Y \rangle \sim \langle u, YX \rangle, \quad \forall u \in \mathcal{G}^*, X, Y \in \mathcal{G}. \tag{A2.26b}$$

Then

$$\langle X \cdot u, Y \rangle \sim \langle u, [Y, X] \rangle = \langle u, YX - XY \rangle \sim \langle -uX + Xu, Y \rangle, \tag{A2.26}$$

so that

$$X \cdot u = Xu - uX. \tag{A2.27}$$

Substituting formulae (A2.25b, (A2.27) into formula (A2.24), we get

$$\langle X\alpha - \alpha X + ua, Y \rangle + \langle v, Xa \rangle, \tag{A2.28}$$

and formula (A2.23) results. ■

Now, formula (A2.25a) yields

$$\begin{aligned}
 \langle Xu, Y \rangle &\sim -\langle u, XY \rangle = -\langle u, \rho^d(X)(Y) \rangle \\
 &= -\langle u, -\rho(X)^\dagger(Y) \rangle \sim \langle \rho(X)(u), Y \rangle,
 \end{aligned} \tag{A2.29}$$

so that

$$Xu = \rho(X)(u). \tag{A2.30}$$

Formula (A2.21) can be now rewritten as

$$\left[\begin{pmatrix} \alpha \\ a \end{pmatrix}, \begin{pmatrix} \beta \\ b \end{pmatrix} \right] = \begin{pmatrix} \rho(a)(\beta) - \rho(b)(\alpha) \\ [a, b] \end{pmatrix}. \tag{A2.31}$$

We see that the Lie algebra structure on $\mathcal{G}^{(1)*}$ is the same as on $\mathcal{G}^{(1)}$. This explains, – and provides an n -dimensional generalization of, – formula (3.17) for the case $\epsilon = \mu = 0$. (Formula (A2.16) is an n -dimensional analog of formula (3.7).)

We conclude by defining the notion of a *symplectic double* of the Lie algebra $\mathcal{G}^{(1)} = \mathcal{G} \ltimes \mathcal{G}^*$ for a quasiassociative \mathcal{G} . It is different from the Drinfel'd's classical double, $\mathcal{G} + \mathcal{G}^*$, discussed in the previous Section.

So, let \mathcal{A} be a quasiassociative ring and $\mathcal{G} = \text{Lie}(\mathcal{A})$. Define $T^*\mathcal{A}$ by the formula [7]

$$\begin{pmatrix} a \\ a^* \end{pmatrix} \begin{pmatrix} b \\ b^* \end{pmatrix} = \begin{pmatrix} ab \\ ab^* \end{pmatrix}, \quad a, b \in \mathcal{A}, \quad a^*, b^* \in \mathcal{A}^*, \tag{A2.32}$$

where the product ab^* is defined by formula (A2.25a). Notice, that $a^*b = 0$ in formula (A2.32), in contrast to the previous definition (A2.25b).

Proposition A2.33. $T^*\mathcal{A}$ (A2.32) is again a quasiassociative algebra.

Proof. We have:

$$\begin{aligned} & \left(\begin{pmatrix} a \\ a^* \end{pmatrix} \begin{pmatrix} b \\ b^* \end{pmatrix} \right) \begin{pmatrix} c \\ c^* \end{pmatrix} - \begin{pmatrix} a \\ a^* \end{pmatrix} \left(\begin{pmatrix} b \\ b^* \end{pmatrix} \begin{pmatrix} c \\ c^* \end{pmatrix} \right) \\ &= \begin{pmatrix} ab \\ ab^* \end{pmatrix} \begin{pmatrix} c \\ c^* \end{pmatrix} - \begin{pmatrix} a \\ a^* \end{pmatrix} \begin{pmatrix} bc \\ bc^* \end{pmatrix} = \begin{pmatrix} (ab)c - a(bc) \\ (ab)c^* - a(bc^*) \end{pmatrix}. \end{aligned} \quad (\text{A2.34})$$

Thus, we need to show that

$$(ab)c^* - a(bc^*) = (ba)c^* - b(ac^*). \quad (\text{A2.35})$$

We have, $\forall d \in \mathcal{A}$:

$$\langle (ab)c^* - a(bc^*), d \rangle \sim \langle c^*, -(ab)d \rangle + \langle bc^*, ad \rangle \sim \langle c^*, -(ab)d - b(ad) \rangle. \quad (\text{A2.36})$$

But

$$(ab)d + b(ad) = (ba)d + a(bd) \quad (\text{A2.37})$$

by formula (A2.9). ■

Since $\text{Lie}(T^*\mathcal{A}) = \text{Lie}(\mathcal{A}) \ltimes [\text{Lie}(\mathcal{A})]^*$:

$$\left[\begin{pmatrix} a \\ a^* \end{pmatrix}, \begin{pmatrix} b \\ b^* \end{pmatrix} \right] = \begin{pmatrix} a \\ a^* \end{pmatrix} \begin{pmatrix} b \\ b^* \end{pmatrix} - \begin{pmatrix} b \\ b^* \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix} = \begin{pmatrix} ab - ba \\ ab^* - ba^* \end{pmatrix}, \quad (\text{A2.38})$$

we can construct a symplectic 2-cocycle, – and thus an \mathcal{O} -operator, – starting with $\mathcal{G}^{(1)} = \mathcal{G} \ltimes \mathcal{G}^*$ rather than \mathcal{G} . This process can be continued indefinitely. It is natural to call the Lie algebra $\mathcal{G}^{(1)} \ltimes \mathcal{G}^{(1)*}$ the *symplectic double* of the Lie algebra $\mathcal{G}^{(1)} = \mathcal{G} \ltimes \mathcal{G}^*$; thus, $\mathcal{G}^{(1)}$ is the symplectic double of \mathcal{G} . According to results of § 6, the space $\text{Fun}(\mathcal{G}^{(1)*})$ carries three compatible Hamiltonian structures: symplectic, linear, and quadratic.

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